

DIAGRAMS WITH SELECTION AND METHOD FOR CONSTRUCTING BOUNDEDLY GENERATED AND BOUNDEDLY SIMPLE GROUPS

ALEXEY MURANOV

Vanderbilt University

April 26, 2004

The existence of an infinite simple boundedly generated 2-generated group and the existence of a boundedly simple 2-generated group containing a free non-cyclic subgroup are proved.

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2000 *Mathematics Subject Classification.* 20E32, 20F05, 20F06.

Key words and phrases. Combinatorial group theory; group presentation; van Kampen diagram; diagram with selection; S-diagram; small cancellation condition; simple group; boundedly simple group; boundedly generated group; bounded generation.

This work was supported in part by the NSF grant DMS 0072307 of Alexander Ol'shanskii and Mark Sapir.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -T $\mathcal{E}\mathcal{X}$

1. INTRODUCTION

Definition. A group G is called *m-boundedly generated* if it has (not necessarily normal) cyclic subgroups C_1, \dots, C_m such that $G = C_1 \cdots C_m$. A group G is called *boundedly generated* if it is m -boundedly generated for some natural m .

Apparently, the definition of bounded generation was motivated by the work [CK] of Carter and Keller, in which they proved that any matrix in $SL_n(\mathcal{O})$, where $n \geq 3$ and \mathcal{O} is the ring of integers of a finite extension of the field of rational numbers, is the product of a bounded number of elementary matrices (bounded depending only on n and \mathcal{O}). Since then, bounded generation has been studied in connection with the *congruence property* (CSP) (see [PR]), *Kazhdan's property* (T) (see [Sh]). The following closely related property of profinite groups is used in the book [LuSe] by Lubotzky and Segal, as well as in [PR]: a profinite group is *boundedly generated* as a profinite group if it is the product of finitely many procyclic subgroups.

Definition. A group G is called *m-boundedly simple* if for any two nontrivial elements $g, h \in G$, the element h may be presented as the product of m or fewer conjugates of $g^{\pm 1}$. A group G is called *boundedly simple* if it is m -boundedly simple for some natural m .

Remark 1.1. For each natural m , the class of m -boundedly simple groups is definable by a formula of the restricted predicate calculus. The class of all simple groups is not.

It is obvious that a boundedly simple group is simple. However, not every simple group is boundedly simple. For example, an infinite alternating group is simple but not boundedly simple.

Every non-abelian simple group coincides with its derived subgroup. In a non-abelian m -boundedly simple group, the derived subgroup in addition has finite commutator width: each element of the derived subgroup may be presented as the product of m or fewer commutators (take an arbitrary nontrivial commutator, every element is the product of m or fewer conjugates of this commutator).

It is not hard to prove the following

Proposition 1.1. *A group is boundedly simple if and only if each of its ultrapowers is simple. If a group is m-boundedly simple, then all its ultrapowers are m-boundedly simple.*

It is not obvious that infinite boundedly simple finitely generated groups or infinite simple boundedly generated groups exist. The following theorem may be found in [O]:

Theorem (S. Ivanov). *For every big enough prime p , there exists a 2-generated infinite group of exponent p , in which there are exactly p distinct conjugacy classes, and therefore, every subgroup of order p has elements from all of these classes.*

It immediately follows that the group whose existence is stated in this theorem is $(p-1)$ -boundedly simple. The original proof of Ivanov's Theorem works only for big p , say, $p \geq 10^{78}$. It heavily uses techniques of graded diagrams. The following two theorems are proved in this paper in a different manner.

Theorem 1. *Let F be a finite-rank free group with a fixed basis. Let S be an infinite subset of F whose all elements are cyclically reduced and are not proper powers. If the symmetrization of S satisfies the small cancellation condition $C'(\lambda)$ for some $\lambda < \frac{1}{13}$, then there exist an infinite simple 2-generated group P and a homomorphism $\phi: F \rightarrow P$ such that ϕ maps S onto P .*

Corollary. *There exists an infinite simple 2-generated group G and 27 elements $x_1, \dots, x_{27} \in G$ such that for every element g of G there exists a natural number n such that $g = x_1^n \cdots x_{27}^n$. Such a group is, in particular, 27-boundedly generated.*

Theorem 2. *There exists a 14-boundedly simple 2-generated group with solvable word problem, containing a free non-cyclic subgroup.*

The purpose of this paper is to develop a method for constructing groups with these kinds of properties. Roughly speaking, the method consists in constructing group presentations by imposing all necessary relations, while making sure that “big parts” of these relations satisfy certain small-cancellation-type conditions. No classical small cancellation conditions are required to hold for the set of imposed relations itself. The main tool for proving that the obtained groups are nontrivial, or even contain free non-cyclic subgroups, is properties of van Kampen *diagrams with selection* that satisfy a certain condition \mathcal{A} with several parameters. Diagrams with selection, also called here *S-diagrams*, are defined and used similarly to maps with partitioned boundaries of cells in [O] and [Sa]. They are diagrams in which some subpaths of the contours of the faces are selected. When choosing a selection on a van Kampen diagram over a constructed group presentation, the subpaths corresponding to the aforementioned “big parts” of the relations are selected. Among other things, the condition \mathcal{A} requires that, with few exceptions, each arc between two faces that lies on selected subpaths of the contours of both faces has to be “short” relative to the lengths of the contours of both (!) faces. (This makes the condition \mathcal{A} essentially different from the condition B in [O].) A condition similar to the condition \mathcal{A} was used by A. Sanin in his paper [Sa] in 1997. The main theorem proved in this paper implies in particular that if a reduced disc S-diagram satisfies the condition \mathcal{A} with values of the parameters satisfying certain inequalities, and the diagram either has “many” faces, or has faces with “long” contours, then the contour of the diagram itself is “long.” This main theorem makes the proofs of Theorems 1 and 2 relatively simple.

2. GRAPHS AND MAPS

Some terms pertaining to graphs and 2-complexes shall be clarified in this section.

If X is a set, then $\|X\|$ shall denote the cardinality of X .

The term *graph* will refer to an object that in graph theory is called an undirected pseudograph (multiple edges and loops are admissible). If Γ is a graph, then $\Gamma(0)$ denotes its set of vertices, and $\Gamma(1)$ denotes its set of edges; $\Gamma(1)$ may be empty, but $\Gamma(0)$ may not. The graph itself is an ordered pair: $\Gamma = (\Gamma(0), \Gamma(1))$. In the context of this paper, *graph* is a synonym of 1-*complex*. A great part of this paper deals with 2-*complexes*. If Φ is a 2-complex, then $\Phi(0)$, $\Phi(1)$, and $\Phi(2)$ denote its set of vertices, its set of edges, and its set of faces (cells) respectively. The 2-complex Φ itself is the ordered triple $(\Phi(0), \Phi(1), \Phi(2))$. Below is a formal definition of combinatorial 0-, 1-, and 2-complexes and their morphisms.

A (combinatorial) 0-complex A is a 1-tuple $(A(0))$ where $A(0)$ is an arbitrary nonempty set. Elements of $A(0)$ are called *vertices* of the complex A .

A 0-complex with exactly 2 vertices will be called a *combinatorial 0-sphere*.

A *morphism* ϕ of a 0-complex A to a 0-complex B is a 1-tuple $(\phi(0))$ where $\phi(0)$ is an arbitrary function $A(0) \rightarrow B(0)$. If A , B , and C are 0-complexes, and $\phi: A \rightarrow B$ and $\psi: B \rightarrow C$ are morphisms, then the product $\psi\phi: A \rightarrow C$ is defined naturally: $(\psi\phi)(0) = \psi(0) \circ \phi(0)$. A morphism $\phi: A \rightarrow B$ is called an *isomorphism* of A with B if there exists a morphism $\psi: B \rightarrow A$ such that $\psi\phi$ is the *identity morphism* of the complex A and $\phi\psi$ is the identity morphism of the complex B .

A (combinatorial) 1-complex A is a 2-tuple (i.e., ordered pair) $(A(0), A(1))$ such that:

- (1) $A(0)$ is an arbitrary nonempty set;
- (2) $A(1)$ is a set of ordered pairs of the form (E, α) where E is a combinatorial 0-sphere and α is a morphism of E to the 0-complex $(A(0))$.

Elements of $A(0)$ are called *vertices* of the complex A , elements of $A(1)$ are called *edges*. The 0-complex $(A(0))$ is called the 0-skeleton of A . All 1-complexes will also be called *graphs*.

A 1-complex representing a circle (e.g., a 1-complex consisting of one vertex and one edge) will be called a *combinatorial 1-sphere*, or a *combinatorial circle*. (The definition of a 1-sphere could be made precise, but then it would become unreasonably long.)

If A and B are 1-complexes, then a *morphism* $\phi: A \rightarrow B$ is a 2-tuple $(\phi(0), \phi(1))$ such that:

- (1) $\phi^0 = (\phi(0))$ is a morphism of the 0-skeleton A^0 of A to the 0-skeleton B^0 of B ;
- (2) $\phi(1)$ is a function on $A(1)$ such that the image of every $e = (E, \alpha) \in A(1)$ under $\phi(1)$ is an ordered pair (e', ξ) where $e' = (E', \alpha') \in B(1)$, ξ is an isomorphism of E with E' , and $\phi^0\alpha = \alpha'\xi$.

Multiplication of morphisms of 1-complexes is defined naturally. For example, if A , B , C are 1-complexes, ϕ and ψ are morphisms, $\phi: A \rightarrow B$, $\psi: B \rightarrow C$, $e = (E, \alpha)$ is an edge of A , $e' = (E', \alpha')$ is an edge of B , $e'' = (E'', \alpha'')$ is an edge of C , $\phi(1)(e) = (e', \xi)$, $\psi(1)(e') = (e'', \zeta)$, then $(\psi\phi)(1)(e) = (e'', \zeta\xi)$ (note that $\zeta\xi$ is an isomorphism of the 0-complex E with the 0-complex E''). *Isomorphisms* of 1-complexes are defined in the usual way.

A (combinatorial) 2-complex A is a 3-tuple $(A(0), A(1), A(2))$ such that:

- (1) $(A(0), A(1))$ is a 1-complex, called the 1-skeleton of A ;
- (2) $A(2)$ is a set of ordered pairs of the form (F, α) where F is a combinatorial 1-sphere and α is a morphism of F to the 1-skeleton of A .

Elements of $A(0)$ are called *vertices* of the complex A , elements of $A(1)$ are called *edges*, elements of $A(2)$ are called *faces*.

If A and B are 2-complexes, then a *morphism* $\phi: A \rightarrow B$ is defined as a 3-tuple $(\phi(0), \phi(1), \phi(2))$ such that:

- (1) $\phi^1 = (\phi(0), \phi(1))$ is a morphism of the 1-skeleton A^1 of A to the 1-skeleton B^1 of B ;

- (2) $\phi(2)$ is a function on $A(2)$ such that the image of every $f = (F, \beta) \in A(2)$ under $\phi(2)$ is an ordered pair (f', ξ) where $f' = (F', \beta') \in B(2)$, ξ is an isomorphism of F with F' , and $\phi^1 \beta = \beta' \xi$.

Products of morphisms of 2-complexes are defined analogously to the case of 1-complexes. The notion of *isomorphism* for 2-complexes is the natural one.

A combinatorial n -complex A is called *finite* if all of the sets $A(0), \dots, A(n)$ are finite. An n -complex A is a *subcomplex* of an m -complex B , $0 \leq n \leq m$, if $A(0) \subset B(0), \dots, A(n) \subset B(n)$. Every morphism ϕ of a combinatorial n -complex A to a combinatorial n -complex B , $n \leq 2$, naturally defines functions $\bar{\phi}^i: A(i) \rightarrow B(i)$, $0 \leq i \leq n$. This notation for these functions associated with a given morphism ϕ shall be used in this section for brevity.

If $e = (E, \alpha)$ is an edge of a graph Γ , then the vertices of Γ that are the images of the vertices of E under $\bar{\alpha}^0$ are called the *end-vertices* of e . An edge is *incident* to its end-vertices. A *loop* is an edge that has only one end-vertex. Two vertices are called *adjacent* if they form the set of end-vertices of some edge. The end-vertex of a loop is adjacent to itself. If v is a vertex of a graph Γ , then the number of all edges of Γ incident to v plus the number of all loops incident to v is called the *degree* of the vertex v and is denoted by $d(v)$ or $d_\Gamma(v)$.

If $f = (F, \beta)$ is a face of a 2-complex Φ , then f is said to be *incident* to the images of the vertices and edges of F under $\bar{\beta}^0$ and $\bar{\beta}^1$ respectively. If the combinatorial circle F has n edges ($\|F(1)\| = n \in \mathbb{N}$), then the number n is called the *degree* of f and is denoted by $d(f)$ or $d_\Phi(f)$.

It is somewhat complicated to talk about combinatorial complexes in purely combinatorial terms. The geometrical intuition may help if together with every combinatorial complex consider some corresponding topological space.

Given a 2-complex Φ , put a 1-point topological space D_v^0 into correspondence to every vertex v of Φ , a topological closed segment D_e^1 to every edge e of Φ , and a topological closed disk D_f^2 to every face f of Φ . Assume that D_x^m and D_y^n are disjoint unless $m = n$ and $x = y$. Consider the topological sum

$$\sum_{v \in \Phi(0)} D_v^0 + \sum_{e \in \Phi(1)} D_e^1 + \sum_{f \in \Phi(2)} D_f^2.$$

Take the quotient of it over an equivalence relation in accordance with the structure of Φ . For example, if $\{v_1, v_2\}$ is the set of end-vertices of an edge e of Φ , then one of the end-vertices of D_e^1 should be identified with the only point of $D_{v_1}^0$, and the other end-vertex should be identified with the only point of $D_{v_2}^0$. Taking the quotient may be done in two steps: first, attach the end-vertices of the segments $\{D_e^1\}_{e \in \Phi(1)}$ to the points of the discrete topological space $\sum_{v \in \Phi(0)} D_v^0$; then, attach the boundaries of the discs $\{D_f^2\}_{f \in \Phi(2)}$ to the obtained “skeleton.” The constructed topological quotient space is defined uniquely up to homeomorphism. This space or any homeomorphic one is called a *topological space of the complex* Φ . It may also be said that Φ *represents* this topological space. Note that the restriction of the quotient function to the interior (in the geometric sense) of every D_e^1 , $e \in \Phi(1)$, and every D_f^2 , $f \in \Phi(2)$, is a homeomorphism onto the image in the quotient space. Every morphism of combinatorial complexes defines some set of continuous functions from a given topological space of the first complex to a given topological space of the second.

From now on, combinatorial and topological languages shall be used together. Moreover, some facts intuitively clear from topological point of view shall be used without proofs.

It is a simple but tiresome task to formulate a combinatorial criterion in terms of the local structure of a 2-complex (*stars* at its vertices) that would determine if a topological space of this complex is a 2-dimensional surface (i.e., a 2-manifold or 2-manifold with a boundary). For an example of constructing surfaces combinatorially using simplicial complexes see [A]. Keeping this in mind, the following definition may be viewed as combinatorial. A *combinatorial surface* is a 2-complex representing some surface. A *combinatorial sphere* and a *combinatorial disc* are 2-complexes representing a sphere and a disc respectively. They play an important role in this paper. Every combinatorial sphere and every combinatorial disc are finite (because of the compactness of spheres and discs). Combinatorial spheres and combinatorial discs may also be defined combinatorially.

Now, a definition of a planar graph may be given easily. A finite graph Γ is *planar* if it is a subgraph of the 1-skeleton of some combinatorial sphere. Actually, every *connected* finite planar graph having at least one edge is the 1-skeleton of some combinatorial sphere.

Most of the estimates in this paper will be based on *Euler's Formula* for a combinatorial sphere: if Φ is a combinatorial sphere, then

$$\|\Phi(0)\| - \|\Phi(1)\| + \|\Phi(2)\| = 2.$$

Using Euler's Formula, it is easy to prove the following well-known

Proposition 2.1. *For any planar graph Γ without loops and without multiple edges, the number of edges of Γ is less than three times the number of its vertices:*

$$\|\Gamma(1)\| < 3\|\Gamma(0)\|.$$

Proof. Without loss of generality, Γ may be assumed to be connected. (It is enough to prove the above inequality for every connected component of Γ .) If $\|\Gamma(0)\| \leq 2$, then the statement is obvious. If $\|\Gamma(0)\| \geq 3$, then Γ is the 1-skeleton of some combinatorial sphere Φ , and the degree of every face of Φ is at least 3. On one hand,

$$2\|\Phi(1)\| = \sum_{f \in \Phi(2)} d(f) \geq 3\|\Phi(2)\|.$$

On the other hand, by Euler's Formula,

$$\|\Phi(0)\| - \|\Phi(1)\| + \|\Phi(2)\| = 2.$$

Therefore,

$$\begin{aligned} 3\|\Phi(0)\| - 3\|\Phi(1)\| + 3\|\Phi(2)\| + 2\|\Phi(1)\| &\geq 6 + 3\|\Phi(2)\|, \\ 3\|\Phi(0)\| &\geq \|\Phi(1)\| + 6, \\ 3\|\Gamma(0)\| &\geq \|\Gamma(1)\| + 6 > \|\Gamma(1)\|. \quad \square \end{aligned}$$

An *orientation* of an edge $e = (E, \alpha)$ of a complex Γ is a total order on the two-element set of vertices of E . There are two possible orientations of every edge,

they are *opposite* to each other. An *oriented edge* is an edge together with one of its orientations. The set of all oriented edges of a complex Γ will be denoted by $\hat{\Gamma}(1)$. Every morphism ϕ of a combinatorial n -complex Γ_1 to a combinatorial n -complex Γ_2 , $n \in \{1, 2\}$, naturally defines a function $\hat{\Gamma}_1(1) \rightarrow \hat{\Gamma}_2(1)$; denote this function by $\hat{\phi}^1$. Two oriented edges obtained from the same edge by picking opposite orientations are *inverse* to each other. The oriented edge inverse to an oriented edge e is denoted by e^{-1} . If e is an oriented edge, (E, α) is a corresponding non-oriented edge, $E(0) = \{v_1, v_2\}$, and v_1 precedes v_2 with respect to the chosen total order on $E(0)$, then $\bar{\alpha}^0(v_1)$ is called the *tail* of e and $\bar{\alpha}^0(v_2)$ is called the *head* of e . An oriented edge *leaves* its tail and *enters* its head. Clearly, the head of an oriented edge is the tail of the inverse oriented edge, and vice versa. For any vertex v in a complex Γ , the number of oriented edges leaving v equals the number of oriented edges entering v and equals the degree of v .

Here follow several definitions related to paths in combinatorial complexes.

A *path* is a finite sequence of alternating vertices and oriented edges such that the following conditions hold: it starts with a vertex and ends with a vertex; the vertex immediately preceding an oriented edge is its tail; the vertex immediately following an oriented edge is its head. The *initial vertex* of a path is its first vertex, the *terminal vertex* of a path is its last vertex, and an *end-vertex* of a path is either its initial or its terminal vertex. A path *starts* at its initial vertex and *ends* at its terminal vertex. The *length* of a path $p = (v_0, e_1, v_1, \dots, e_n, v_n)$ is n ; it is denoted by $|p|$. The vertices v_1, \dots, v_{n-1} of this path are called its *intermediate* vertices. A *trivial path* is a path of length zero. By abuse of notation, a path of the form (v_1, e, v_2) , where e is an oriented edge from v_1 to v_2 , shall be denoted by e , and a trivial path (v) shall be denoted by v .

The *inverse path* to a path p is defined naturally and is denoted by p^{-1} . If the terminal vertex of a path p_1 coincides with the initial vertex of a path p_2 , then the *product* $p_1 p_2$ is defined naturally. A path s is called an *initial subpath* of a path p if $p = sq$ for some path q . A path s is called a *terminal subpath* of a path p if $p = qs$ for some path q .

A *cyclic path* is a path such that its terminal vertex coincides with its initial vertex. A *cycle* is the set of all *cyclic shifts* of some cyclic path. The *length* of a cycle c , denoted by $|c|$, is the length of an arbitrary representative of c . A *trivial cycle* is a cycle of length zero. A path p is a *subpath of a cycle* c if for some representative r of c and for some natural n , p is a subpath of r^n (i.e., of the product of n copies of r). Let c be a cycle in which no oriented edge occurs more than once. Then a set of paths S is said to *cover* c if all the elements of S are nontrivial subpaths of c , and every oriented edge that occurs in c also occurs in some path from S .

A path is *reduced* if it does not have a subpath of the form ee^{-1} where e is an oriented edge. A cyclic path is *cyclically reduced* if it is reduced and its first oriented edge is not inverse to its last oriented edge. (For example, all trivial paths are cyclically reduced.) A cycle is *reduced* if it consists of cyclically reduced cyclic paths. A path is *simple* if it is nontrivial, reduced, and none of its intermediate vertices appears in it more than once. A cycle is *simple* if it consists of simple cyclic paths.

All paths in any graph are naturally partially ordered by the relation “is a subpath of.” A path is called *maximal* in some set of paths if it is maximal in this

set with respect to the above order (i.e., if this path is not a proper subpath of any other path in this set).

Every morphism ϕ of combinatorial complexes naturally defines a function from the set of paths (respectively cycles) of the first complex to the set of paths (respectively cycles) of the second. The image of a given path or cycle under this function will be referred to as the *image relative to ϕ* , or *ϕ -image*.

An *oriented arc* in a complex Γ is a simple path all intermediate vertices of which have degree 2 in Γ . A *non-oriented arc*, or simply *arc*, may be defined as a pair of mutually inverse oriented arcs. The *length* of a non-oriented arc u is defined as the length of either of the two associated oriented arcs and is denoted by $|u|$. The concepts of a *subarc*, a *maximal arc*, and so on, are self-explanatory. Sometimes edges will be viewed as arcs, and oriented edges as oriented arcs. The *set of edges* of an oriented arc (respectively of an arc) is the set of all edges that with some orientation occur in that oriented arc (respectively in either of the two corresponding oriented arcs). An *intermediate vertex* of an arc is an intermediate vertex of either of the corresponding oriented arcs. Two arcs, or oriented arcs, are said to *overlap* if their sets of edges are not disjoint. A set of arcs, or oriented arcs, is called *non-overlapping* if any two distinct elements of this set have disjoint sets of edges. An arc (or an edge) u *lies on* a path p if at least one of the oriented arcs (respectively oriented edges) associated with u is a subpath of p . A set of arcs A *covers* a set of arcs (or edges) B if every element of B is a subarc of some element of A .

Combinatorial complexes representing orientable surfaces may be oriented in a combinatorial manner. To define the combinatorial notion of *orientation*, consider first an arbitrary combinatorial circle Γ . Consider a function that for every edge e of Γ chooses an orientation of e . Call an edge with a chosen orientation a *chosen oriented edge*. Say that such a function chooses *coherent orientations* if every vertex of Γ is the tail of exactly one of the chosen oriented edges and the head of exactly one of the chosen oriented edges. An *orientation* of a combinatorial circle is a function on the set of its edges that chooses coherent orientations of the edges. An *oriented combinatorial circle* is a combinatorial circle together with one of its two possible orientations. Note that in an oriented combinatorial circle Γ there is a unique simple cycle whose oriented edges are the chosen ones. Call this cycle the *chosen cycle* of Γ .

Now, consider a face $f = (F, \beta)$ of a 2-complex Φ . An *orientation* of f is an orientation of the combinatorial circle F . An *oriented face* is a face together with one of its orientations. The set of all oriented faces of a complex Φ will be denoted by $\hat{\Phi}(2)$. (Note that $\|\hat{\Phi}(2)\| = 2\|\Phi(2)\|$.) Every morphism ϕ of a combinatorial 2-complex Φ_1 to a combinatorial 2-complex Φ_2 naturally defines a function $\hat{\Phi}_1(2) \rightarrow \hat{\Phi}_2(2)$; denote this function by $\hat{\phi}^2$. Two oriented faces obtained from the same face by picking *opposite* orientations are *inverse* to each other. The oriented face inverse to an oriented face f is denoted by f^{-1} . Each oriented face has a uniquely defined *boundary cycle*: if f is an oriented face of Φ and (F, β) is its underlying non-oriented face, then the boundary cycle of f is the cycle in Φ which is the β -image of the chosen cycle of F . Boundary cycles of mutually inverse oriented faces are mutually inverse; they are also called *boundary cycles* of the corresponding non-oriented face.

Consider now a combinatorial surface Φ . Consider a function θ that for every face f of Φ chooses an orientation of f . Call a face with a chosen orientation a

chosen oriented face. For every oriented edge e of Φ , count the number of all such ordered pairs (f, x) that $f = (F, \beta)$ is a face of Φ , x is an oriented edge of F , x is chosen with respect to the orientation $\theta(f)$ of F , and $e = \hat{\beta}^1(x)$ (in particular, f is incident to e). The function θ is said to choose *coherent orientations* if for every oriented edge of Φ the number defined above is either 1 or 0. An *orientation* of a combinatorial surface is a choice of coherent orientations of all of its faces. A combinatorial surface that admits orientation is called *orientable*. The fact of the matter is that a combinatorial surface is orientable if and only if it represents an orientable surface, but this fact is not used in the paper. There exist exactly two orientations of any connected orientable combinatorial surface. The *contour* of a (non-oriented) face in an oriented combinatorial surface is the boundary cycle of the corresponding chosen oriented face. An *isomorphism* of oriented combinatorial surfaces A and B is an isomorphism of A and B as 2-complexes which additionally preserves the chosen orientations of the faces.

The only 2-complexes used in this paper are subcomplexes of combinatorial spheres. Connected components of such subcomplexes with some additional structure will be called *maps*.

Definition. A *spherical map* is an oriented combinatorial sphere.

Consider an oriented combinatorial sphere Δ . By removing some of its faces, one can obtain a subcomplex Φ of the underlying 2-complex of Δ . Note that such a subcomplex Φ is connected (Φ and Δ have the same 1-skeleton). If the contours of the removed faces and the orientations of the faces of Φ induced from Δ are known, then the oriented combinatorial sphere Δ may be reconstructed from Φ up to isomorphism. (Just attach new faces to cover the “holes” in Φ and orient all faces appropriately.)

Definition. A *nontrivial map* Φ is any 2-complex that can be obtained from some oriented combinatorial sphere Δ in the way described above, together with the inherited orientation of its faces and with the set of the contours of the removed faces of Δ , called the *contours* of Φ . Such a spherical map Δ is called a *spherical closure* of the given nontrivial map Φ . If Φ is a nontrivial map and Δ is its spherical closure, then those faces of Δ that are in the same time faces of Φ are called *proper*, and the other ones are called *improper*. A *trivial map* is a 2-complex consisting of a single vertex together with the one-element set consisting of the only (trivial) cycle in this 2-complex. This (trivial) cycle is called the *contour* of this trivial map. A map is called *degenerate* if its set of faces is empty.

Remark 2.1. At this point, one may want to redefine a spherical map as an oriented combinatorial sphere together with the empty set (of its contours).

The following definition is used to distinguish “pathological” maps.

Definition. An *exceptional map* is a spherical map, the 1-skeleton of whose underlying 2-complex is a combinatorial circle (such a map must consist of two faces “glued” together along their boundary cycles).

In a non-exceptional spherical map, any two distinct maximal arcs do not overlap.

Any connected subcomplex of any map naturally inherits a map structure (even if it was obtained by removing not only faces, but also edges and vertices).

Definition. A *submap* of a map is a connected subcomplex with the inherited map structure.

Note: in general, a spherical map Δ is not a spherical closure of a given nontrivial submap of Δ .

In every map, each oriented edge appears either in the contour of some face of the map or in a contour of the map.

Definition. A map is *simple* if all of its contours are simple cycles without common vertices.

Remark 2.2. If a map Δ is simple, then every edge of Δ is incident with some face of Δ .

Definition. A *disc map* is a map with exactly one contour.

Remark 2.3. A disc map is simple if and only if its underlying 2-complex is a combinatorial disc.

The contours of faces in a disc map may be thought of as *oriented counterclockwise*, and the contour of the map itself may be thought of as *oriented clockwise* (this convention corresponds to the way a disc map is usually pictured). Divide the set of all simple cycles in a disc map into *oriented clockwise* and *oriented counterclockwise* accordingly.

An oriented arc u of a map Φ is *incident* to a face f of Φ if it is a subpath of one of the boundary cycles of f . An arc is *incident* to f if one/both of the corresponding oriented arcs are incident to f . An arc incident to a face f is also called an *arc* of f . An arc which is incident to some face is either *internal* (does not lie on any of the contours of the map) or *external* (lies on some contour of the map). Internal oriented arcs divide into *inter-facial* (incident to two different faces) and *intro-facial* (incident to only one face). An internal arc u is *between* faces f_1 and f_2 if $\{f_1, f_2\}$ is the set of all faces incident to u . Every edge may be considered as an arc. Therefore, it makes sense to say that an edge incident to some face is internal, external, inter-facial, or intro-facial. An oriented arc or an oriented edge is internal, external, inter-facial or intro-facial if the corresponding non-oriented arc or edge is such. Every intro-facial oriented arc u in a disc map is either *outward* (u is an initial subpath of some simple path with the terminal vertex on the contour of the map), or *inward* (u^{-1} is outward).

Definition. Let Δ be a map, c be a nontrivial cycle in Δ , Δ' be a simple disc map. Say that c *cuts* Δ' out of Δ if there exists a morphism ζ of the underlying 2-complex of Δ' to the underlying 2-complex of Δ preserving the chosen orientations of the faces such that $\bar{\zeta}^2: \Delta'(2) \rightarrow \Delta(2)$ is injective and c is the ζ -image of the contour of Δ' . Call such a morphism ζ a *pasting morphism*.

Remark 2.4. If a cycle c cuts a simple disc map Δ' out of a map Δ , then Δ' is essentially determined by Δ and c .

It is also common to call a simple disc map that is cut out of a given map Δ just a “simple disc submap” of Δ , but it is not compatible with the definition of a submap given in this paper.

If Δ_0 is a simple disc submap of a map Δ , then the contour of Δ_0 cuts Δ_0 out of Δ , and the natural morphism of Δ_0 to Δ is the corresponding pasting morphism. In a disc map, any simple cycle oriented clockwise is the contour of some simple disc submap; therefore, it cuts out that simple disc submap.

Let Δ be a combinatorial surface or a subcomplex of a combinatorial surface. Call two distinct faces of Δ *contiguous* if there exist an edge incident with both of

them. (The notion of contiguity will only be used for distinct faces.) Let F be a nonempty set of faces of Δ . Let S be the set of all (non-ordered) pairs of distinct contiguous faces in the set F . Clearly, there exist a graph Γ and bijections

$$\alpha_0: F \rightarrow \Gamma(0), \quad \alpha_1: S \rightarrow \Gamma(1),$$

such that a face $f \in F$ belongs to a pair $P \in S$ if and only if the vertex $\alpha_0(f)$ is incident to the edge $\alpha_1(P)$. Indeed, choose vertices of Γ in bijective correspondence with faces of the set F , and connect any two vertices that correspond to distinct contiguous faces with an edge.

Definition. Such a graph Γ is called a *contiguity graph* for the set F in Δ .

Remark 2.5. A contiguity graph is unique up to graph isomorphism. Obviously, any contiguity graph cannot have loops or multiple edges. A contiguity graph of the set of all faces in a combinatorial surface is connected. It is easy to see that a contiguity graph of any nonempty set of faces in (any subcomplex of) a combinatorial sphere is a planar graph.

Consider a combinatorial sphere Δ . Let E be a nonempty non-overlapping set of arcs of Δ satisfying the following condition: for any two distinct elements e_1, e_2 of E , there exists a finite sequence (x_0, \dots, x_n) of elements of E such that $x_0 = e_1$, $x_n = e_2$, and for every $i = 0, \dots, n-1$, the arcs x_i and x_{i+1} are both incident to some face of Δ . Let F be the set of all faces of Δ that are incident to at least one element of E . Let Ψ be the 2-complex obtained from Δ by removing all the faces that belong to F , and all the edges and all the intermediate vertices of all the arcs that belong to E . Note that all the end-vertices of all the arcs from E are vertices of Ψ . Let V be the set of all connected components of Ψ . Let \hat{E} be the set of all oriented arcs associated with arcs from E .

Definition. An *estimating 2-complex* for the set E in the combinatorial sphere Δ is a combinatorial sphere Φ such that there exist bijections

$$\alpha_0: V \rightarrow \Phi(0), \quad \alpha'_1: \hat{E} \rightarrow \hat{\Phi}(1), \quad \alpha_2: F \rightarrow \Phi(2),$$

satisfying the following properties:

- The bijection α'_1 takes mutually inverse oriented arcs of Δ to mutually inverse oriented edges of Φ .
- For each $v \in V$, $e \in \hat{E}$, the vertex $\alpha_0(v)$ is the tail (respectively head) of the oriented edge $\alpha'_1(e)$ if and only if the initial vertex (respectively terminal vertex) of the oriented arc e belongs to the component v .
- If one of the boundary cycles of $f \in F$ has a representative of the form

$$x_0 y_1 x_1 \cdots y_n x_n,$$

where each path x_i is wholly contained in some $v_i \in V$, and $y_1, \dots, y_n \in \hat{E}$, then

$$(\alpha_0(v_0), \alpha'_1(y_1), \alpha_0(v_1), \dots, \alpha'_1(y_n), \alpha_0(v_n))$$

is a representative of a boundary cycle of the face $\alpha_2(f)$.

Remark 2.6. It may be shown under the above assumptions that an estimating 2-complex exists and is unique up to isomorphism.

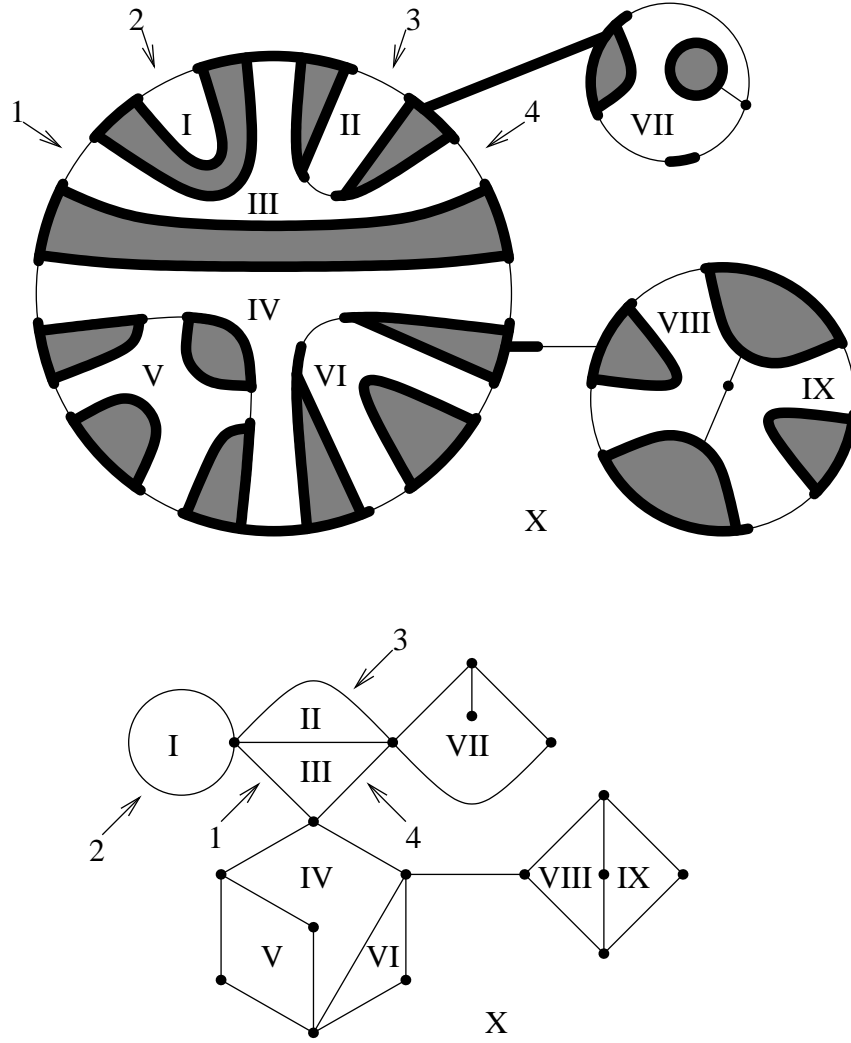


FIG. 1: COMBINATORIAL SPHERE AND ESTIMATING 2-COMPLEX.

Informally speaking, an estimating 2-complex for the set E in Δ may be obtained as follows. First, take all the edges that do not belong to elements of E and all the faces that are not incident to elements of E and collapse them into points. These points will be some of the vertices of the estimating 2-complex. Second, erase all the intermediate vertices from the elements of E so that every arc from E becomes an edge. The condition imposed on E ensures that the obtained 2-complex is a combinatorial sphere.

Figure 1 shows an example of a combinatorial sphere Δ and an estimating 2-complex Φ for some non-overlapping set of arcs E in Δ . The upper part of the picture shows a stereographic projection of Δ , in which the elements of E are indicated by the thinner lines. The lower part shows a stereographic projection of Φ . The roman numerals indicate the correspondence between the faces of Δ incident with the arcs from E (10 faces) and the faces of Φ . The arabic numerals indicate the correspondence between some arcs from E and the corresponding edges of Φ (the total number of elements in E is 25). The parts of Δ that are to be “collapsed”

are shown as shaded regions and thicker lines (there are 15 such “parts”). There are also 2 vertices in Φ that correspond to vertices of Δ .

The estimations used in the proof of the main theorem are based on Euler’s Formula applied to estimating 2-complexes. Objects similar to estimating 2-complexes were extensively used in [O]; there they were called *estimating graphs*.

3. MAPS WITH SELECTION

The proofs of Theorems 1 and 2 given in the end of this paper are based on constructing group presentations such that some sets of subwords of the defining relations satisfy certain small-cancellation-type conditions, though the presentations themselves do not need to satisfy classical small cancellation conditions. This is why it is convenient to consider maps and van Kampen diagrams together with some systems of selected subpaths of the contours of their faces.

Definition. A *selection* on a face f of a map Δ is a set of nontrivial reduced subpaths of the contour of f such that for each path in this set, all of its nontrivial subpaths belong to the set as well (i.e., the set is closed under taking nontrivial subpaths). A *selection* on a map Δ is a set of nontrivial reduced subpaths of the contours of faces of Δ which is closed under taking nontrivial subpaths. An *S-map*, or *map with selection*, is a map together with a selection on it.

A path in an S-map is *selected* if it belongs to the selection. An oriented edge in an S-map is *selected* if it is selected as a path. A path or an oriented edge is *double-selected* if it is selected along with its inverse. An edge or an arc is *double-selected* if it is internal and both of the corresponding oriented edges or arcs are selected. An external edge is *selected* if one of the associated oriented edges is selected.

If ζ is a morphism of the underlying 2-complex of a map Δ_1 to the underlying 2-complex of a map Δ_2 , and ζ preserves the chosen orientations of the faces, then any selection on Δ_2 naturally induces a selection on Δ_1 via ζ : a path p in Δ_1 is selected if and only if its ζ -image in Δ_2 is selected (note that if the ζ -image of a path is reduced, then the path itself is reduced). Say that a cycle c *cuts* a simple disc S-map Δ' out of an S-map Δ if c cuts the underlying map of Δ' out of the underlying map of Δ , and the selection on Δ' is the one induced from Δ via some pasting morphism. Only those pasting morphisms of the underlying map of Δ' that induce the given selection on Δ' are called *pasting morphisms* of the S-map Δ' to Δ .

Any submap of an S-map has a naturally induced selection. Say that an S-map Δ_0 is an *S-submap* of an S-map Δ if Δ_0 is a submap of Δ together with the induced selection.

By abuse of notation, S-maps sometimes will be called just maps.

Definition. An S-map Δ is said to satisfy the *condition* $\mathcal{Z}(n)$, $n \in \mathbb{N} \cup \{0\}$, if for every set S of selected paths that covers the contour of some simple disc submap of Δ , the set S has more than n maximal elements; in particular, S has at least $n + 1$ element.

Proposition 3.1. *Let Φ be a non-degenerate disc map. Suppose a set S consists of n reduced paths and covers the contour of Φ . Then there exists a maximal simple disc submap Φ_1 of Φ whose contour may be covered by n or fewer subpaths of the paths comprising S .*

Proof. Let c be the result of cyclic reduction of the contour of Φ . Since Φ is a non-degenerate disc map, the cycle c is the contour of a non-degenerate disc map Φ' ; in particular, c is nontrivial. Let $q_1, \dots, q_{n'}$ be such subpaths of c that the product $q_1 \cdots q_{n'}$ is a cyclic path representing c , each q_i is a nontrivial subpath of some element of S , and each element of S has at most one of the paths $q_1, \dots, q_{n'}$ as a subpath. It is not hard to prove that such paths $q_1, \dots, q_{n'}$ exist. Note that $n' \leq n$.

If Φ' is simple, then $\Phi_1 = \Phi'$ is a desired submap. Consider the case when Φ' is not simple. Since the map Φ' is non-degenerate, it has two distinct maximal simple disc submaps Φ_1 and Φ_2 , whose contours have representatives which are subpaths of the contour of Φ' . Let p_i be the representative of the contour of Φ_i which is a subpath of the contour of Φ' , $i = 1, 2$. Let v_i be the initial vertex of q_i , $i = 1, \dots, n'$. Let w_i be the initial (and the terminal) vertex of p_i , $i = 1, 2$. Let k_i be the number of vertices from the set $\{v_1, \dots, v_{n'}, w_1, w_2\}$ which occur in the contour of Φ_i , $i = 1, 2$. It is easy to see that either $k_1 \leq n'$ or $k_2 \leq n'$. For the sake of definiteness, assume that $k_1 \leq n'$. Then the contour of Φ_1 has a representative of the form $q_1^{(1)} \cdots q_{k_1}^{(1)}$ where every $q_i^{(1)}$ is a nontrivial subpath of some q_j . The set $\{q_1^{(1)}, \dots, q_{k_1}^{(1)}\}$ covers the contour of Φ_1 . \square

Corollary. *Let Δ be an S -map. Suppose the contour of some non-degenerate disc submap of Δ is covered by a set of n or fewer selected paths. Then Δ does not satisfy the condition $\mathcal{Z}(n)$.*

4. CONDITION \mathcal{A}

Roughly speaking, the condition \mathcal{A} defined in this section is a generalization of the small cancellation condition C' formulated in terms of the underlying maps of reduced van Kampen diagrams, rather than in terms of defining relations. (About the condition C' and van Kampen diagrams, see Section 11.)

Recall that the degree $d(f)$ of a face f of a map is defined as the length of the contour of f .

Definition. Let Δ be an S -map. Let $\bar{a} = (k_1, k_2, k_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where

$$k_1, k_2, k_3: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4: \Delta(2) \rightarrow [0, 1].$$

The S -map Δ is said to satisfy the *condition $\mathcal{A}(\bar{a})$* if it satisfies the following seven conditions:

- $\mathcal{A}_1(k_1)$ For each face f of Δ , there exists at least one selected subpath of the contour of f , and the number of maximal selected subpaths of the contour of f does not exceed $k_1(f)$ (note that if all nontrivial subpaths are selected, then there is no maximal selected subpath).
- $\mathcal{A}_2(k_2)$ For each face f of Δ , there are at most $k_2(f)$ subpaths of the contour of f that are of the form $p_1 q p_2$ where p_1 and p_2 are maximal selected paths, q is a nontrivial path, and no subpath of q is selected.
- $\mathcal{A}_3(k_3)$ For each face f of Δ , there are at most $k_3(f)$ subpaths of the contour of f that are of the form $p_1 p_2$ where p_1 and p_2 are maximal selected paths, and $p_1 p_2$ is not reduced.
- $\mathcal{A}_4(\lambda_1)$ For each face f of Δ , if S is the number of non-selected oriented edges in the contour of f , then

$$S \leq \lambda_1(f) d(f).$$

$\mathcal{A}_5(\lambda_2)$ For each face f of Δ , if u is a double-selected intro-facial arc incident to f , then

$$|u| \leq \lambda_2(f)d(f).$$

$\mathcal{A}_6(\lambda_2, \lambda_3)$ For every two distinct faces f_1, f_2 of Δ , if U is a non-overlapping set of double-selected arcs between f_1 and f_2 which covers the set of all double-selected edges between f_1 and f_2 , then

$$\sum_{u \in U} |u| \leq \|U\| \cdot \min\{\lambda_2(f_1)d(f_1), \lambda_2(f_2)d(f_2)\} + \min\{\lambda_3(f_1)d(f_1), \lambda_3(f_2)d(f_2)\}.$$

$\mathcal{A}_7(\lambda_2, \lambda_4)$ For each face f of Δ , if p is a simple selected subpath of the contour of a distinct face of Δ , U is a non-overlapping set of double-selected arcs incident to f and lying on p , and U covers the set of all double-selected edges that are incident to f and lie on p , then

$$\sum_{u \in U} |u| \leq (\|U\|\lambda_2(f) + \lambda_4(f))d(f).$$

In particular, if a double-selected inter-facial arc u is incident to a face f , then

$$|u| \leq (\lambda_2(f) + \lambda_4(f))d(f).$$

A map Δ is said to satisfy the condition $\mathcal{A}(\bar{a})$ if there exists a selection on Δ such that Δ with this selection satisfies $\mathcal{A}(\bar{a})$.

Remark 4.1. If for each face f of Δ , the length of every double-selected arc incident to f is at most $\lambda_2(f)d(f)$, then Δ automatically satisfies the conditions $\mathcal{A}_5(\lambda_2)$, $\mathcal{A}_6(\lambda_2, \lambda_3)$, $\mathcal{A}_7(\lambda_2, \lambda_4)$. If Δ satisfies $\mathcal{A}_1(k_1)$, then it also satisfies $\mathcal{A}_2(k_1)$ and $\mathcal{A}_3(k_1)$.

If Δ_1 and Δ_2 are two maps, ζ is a morphism from Δ_1 to Δ_2 which preserves the chosen orientations of the faces, and

$$k_1, k_2, k_3: \Delta_2(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4: \Delta_2(2) \rightarrow [0, 1], \\ \bar{a} = (k_1, k_2, k_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

then define

$$k_1.\zeta = k_1 \circ \bar{\zeta}^2, \quad k_2.\zeta = k_2 \circ \bar{\zeta}^2, \quad k_3.\zeta = k_3 \circ \bar{\zeta}^2, \\ \lambda_1.\zeta = \lambda_1 \circ \bar{\zeta}^2, \quad \lambda_2.\zeta = \lambda_2 \circ \bar{\zeta}^2, \quad \lambda_3.\zeta = \lambda_3 \circ \bar{\zeta}^2, \quad \lambda_4.\zeta = \lambda_4 \circ \bar{\zeta}^2, \\ \bar{a}.\zeta = (k_1.\zeta, k_2.\zeta, k_3.\zeta; \lambda_1.\zeta, \lambda_2.\zeta, \lambda_3.\zeta, \lambda_4.\zeta),$$

where $\bar{\zeta}^2$ is the function $\Delta_1(2) \rightarrow \Delta_2(2)$ associated with ζ .

Remark 4.2. If an S-map Δ satisfies the condition $\mathcal{A}(\bar{a})$, a simple disc S-map Δ' is cut out of Δ , and ζ is a pasting morphism, then the S-map Δ' satisfies the condition $\mathcal{A}(\bar{a}.\zeta)$. A similar statement is true for each of the conditions \mathcal{A}_1 – \mathcal{A}_7 separately.

5. ESTIMATING LEMMAS

This section contains two lemmas used in the proof of the main theorem for estimating the total number of double-selected internal edges of an S-map. The proofs of these lemmas use one well-known combinatorial result published by Philip Hall in 1935 [H].

If X is a set, then $\mathcal{P}(X)$ shall denote the set of all subsets of X (the power set of X).

Lemma (P. Hall, 1935). *Let A and B be two finite sets. Let f be a function from A to $\mathcal{P}(B)$. Let a function $F: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be defined by $F(X) = \bigcup_{x \in X} f(x)$. Then the following are equivalent:*

- (a) *There exists an injection $h: A \rightarrow B$ such that for all $x \in A$, $h(x) \in f(x)$.*
- (b) *For each subset X of A , $\|X\| \leq \|F(X)\|$.*

The proof of this fact given here appears to be more concise than the original one:

Proof. The implication (a) \Rightarrow (b) is obvious. To prove the converse implication, induct on $\|A\|$.

If $\|A\| = 0$, then $A = \emptyset$ and the conclusion of this lemma is obvious (take $h = \emptyset$).

Let n be a natural number. Suppose the implication (a) \Leftarrow (b) holds under the additional assumption that $\|A\| < n$. Now, assume that $\|A\| = n$ and suppose that (b) holds.

Case 1: there exists a proper nonempty subset A_1 of the set A such that

$$\|F(A_1)\| = \|A_1\|.$$

Then let $B_1 = F(A_1)$, $A_2 = A \setminus A_1$, $B_2 = B \setminus B_1$. Note that $\|A_1\| < n$ and $\|A_2\| < n$. Let $f_1 = f|_{A_1}$, $F_1 = F|_{A_1}$. Note that $f_1: A_1 \rightarrow \mathcal{P}(B_1)$ and $F_1: \mathcal{P}(A_1) \rightarrow \mathcal{P}(B_1)$. Let $f_2: A_2 \rightarrow \mathcal{P}(B_2)$, $F_2: \mathcal{P}(A_2) \rightarrow \mathcal{P}(B_2)$ be defined by

$$f_2(x) = f(x) \cap B_2, \quad F_2(X) = F(X) \cap B_2.$$

Obviously,

$$(\forall X \subset A_1) (\|X\| \leq \|F_1(X)\|).$$

It is also easy to see that

$$(\forall X \subset A_2) (\|X\| \leq \|F_2(X)\|).$$

Therefore, by the inductive assumption, there exist injections $h_1: A_1 \rightarrow B_1$ and $h_2: A_2 \rightarrow B_2$ such that

$$(\forall x \in A_1) (h_1(x) \in f_1(x)) \quad \text{and} \quad (\forall x \in A_2) (h_2(x) \in f_2(x)).$$

Clearly, $h = h_1 \cup h_2$ is an injection $A \rightarrow B$ such that

$$(\forall x \in A) (h(x) \in f(x)).$$

Case 2: for each proper nonempty subset A_1 of the set A ,

$$\|F(A_1)\| \geq \|A_1\| + 1.$$

Then take an arbitrary $x_0 \in A$ and an arbitrary $y_0 \in f(x_0)$. Let $A_1 = A \setminus \{x_0\}$, $B_1 = B \setminus \{y_0\}$. Note that $\|A_1\| = n - 1$. Let $f_1: A_1 \rightarrow \mathcal{P}(B_1)$, $F_1: \mathcal{P}(A_1) \rightarrow \mathcal{P}(B_1)$ be defined by

$$f_1(x) = f(x) \setminus \{y_0\}, \quad F_1(X) = F(X) \setminus \{y_0\}.$$

It is obvious that

$$(\forall X \subset A_1)(\|X\| \leq \|F_1(X)\|).$$

Therefore, by the inductive assumption, there exists an injection $h_1: A_1 \rightarrow B_1$ such that

$$(\forall x \in A_1)(h_1(x) \in f_1(x)).$$

Let $h: A \rightarrow B$ be the extension of h_1 to the whole of A by putting $h(x_0) = y_0$. Clearly, h is an injection satisfying

$$(\forall x \in A)(h(x) \in f(x)).$$

Since either Case 1 or Case 2 must take place, the implication $(a) \Leftarrow (b)$ holds when $\|A\| = n$. The inductive step is done. \square

Corollary. *Let A and B be two finite sets. Let f be a function from A to $\mathcal{P}(B)$ and let w be a function from B to $\mathbb{N} \cup \{0\}$. Let a function $F: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be defined by $F(X) = \bigcup_{x \in X} f(x)$. Then the following are equivalent:*

- (a) *There exists a function $h: A \rightarrow B$ such that:*
 - (1) *for all $x \in A$, $h(x) \in f(x)$, and*
 - (2) *for each $y \in B$, the full pre-image of y under h consists of at most $w(y)$ elements.*
- (b) *For each subset X of A , $\|X\| \leq \sum_{y \in F(X)} w(y)$.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. To prove the converse implication, consider the set B' and functions $f': A \rightarrow \mathcal{P}(B')$ and $F': \mathcal{P}(A) \rightarrow \mathcal{P}(B')$ defined by the formulae

$$\begin{aligned} B' &= \{ (b, n) \mid b \in B, n \in \mathbb{N} \cup \{0\}, n < w(b) \}, \\ f'(x) &= \{ (b, n) \mid b \in f(x), n \in \mathbb{N} \cup \{0\}, n < w(b) \}, \\ F'(X) &= \bigcup \{ f'(x) \mid x \in X \}. \end{aligned}$$

Since

$$(\forall X \subset A)(\|F'(X)\| = \sum_{y \in F(X)} w(y)),$$

it follows from the previous lemma that there exists an injection $h': A \rightarrow B'$ such that

$$(\forall x \in A)(h'(x) \in f'(x)).$$

Let $h = p_1 \circ h'$ where p_1 is the first projection from $B \times (\mathbb{N} \cup \{0\})$ onto B . The function h is a desired one. \square

Estimating Lemma 1. *Let Δ be an S -map with $c \leq 3$ contours satisfying the condition $\mathcal{Z}(2)$. Let k_1, k_2, k_3 be functions $\Delta(2) \rightarrow \mathbb{N} \cup \{0\}$. Suppose Δ satisfies the conditions $\mathcal{A}_1(k_1), \mathcal{A}_2(k_2), \mathcal{A}_3(k_3)$. Suppose Δ has at least one double-selected internal edge. Let U be a non-overlapping set of double-selected internal arcs of Δ covering the set of all double-selected internal edges of Δ , with the minimal possible number of elements. Then*

$$\|U\| \leq \sum_{f \in \Delta(2)} (3 + k_1(f) + k_2(f) + k_3(f)) + 2c - 6,$$

and there exists a function $h: U \rightarrow \Delta(2)$ such that:

- (1) each arc $u \in U$ is incident to the face $h(u)$, and
- (2) the full pre-image of each face $f \in \Delta(2)$ under h consists of at most $3 + k_1(f) + k_2(f) + k_3(f)$ elements.

Proof. For each $x \in U$, let $g(x)$ be the set of all faces of Δ incident to u . For each $X \subset U$, let $G(X) = \bigcup_{x \in X} g(x)$. Take an arbitrary nonempty subset E of U . It is to be proved that

$$\|E\| \leq \sum_{f \in G(E)} (3 + k_1(f) + k_2(f) + k_3(f)) + 2c - 6.$$

Let $\bar{\Delta}$ be a spherical closure of Δ . Without loss of generality, assume that E cannot be split into the disjoint union of two nonempty subsets E_1 and E_2 so that $G(E_1) \cap G(E_2) = \emptyset$. Then there exists an estimating 2-complex Φ for E in $\bar{\Delta}$.

Denote $G(E)$ by F . Let V be the set of all connected components of the 2-complex obtained from $\bar{\Delta}$ by removing all the faces that belong to F , and all the edges and intermediate vertices of all the arcs that belong to E . Let W denote the set of all components of this complex that contain improper faces of $\bar{\Delta}$. Note that $\|W\| \leq c$. Let \hat{E} be the set of all oriented arcs associated with arcs from E . Let

$$\alpha_0: V \rightarrow \Phi(0), \quad \alpha'_1: \hat{E} \rightarrow \hat{\Phi}(1), \quad \alpha_2: F \rightarrow \Phi(2)$$

be such bijections as in the definition of an estimating 2-complex. Let W' be the α_0 -image of the set W .

Consider Φ . Let $n_0 = \|\Phi(0)\| = \|V\|$, $n_1 = \|\Phi(1)\| = \|E\|$, $n_2 = \|\Phi(2)\| = \|F\|$. Let $l = \|W'\| = \|W\|$. Consider the vertices of Φ that are not in W' . Let $m_0^{(1)}$ be the number of such vertices of degree 1, $m_0^{(2)}$ be the number of such vertices of degree 2, $m_0^{(\geq 3)}$ be the number of such vertices of degree at least 3.

On one hand, by Euler's Formula,

$$n_0 - n_1 + n_2 = 2;$$

on the other hand,

$$2n_1 = \sum_{x \in \Phi(0)} d_\Phi(x) \geq m_0^{(1)} + 2m_0^{(2)} + 3m_0^{(\geq 3)} + \sum_{w' \in W'} d_\Phi(w').$$

Therefore,

$$\begin{aligned} 3n_0 - 3n_1 + 3n_2 + 2n_1 &\geq 6 + m_0^{(1)} + 2m_0^{(2)} + 3m_0^{(\geq 3)} + l, \\ 3m_0^{(1)} + 3m_0^{(2)} + 3m_0^{(\geq 3)} + 3l - n_1 + 3n_2 &\geq 6 + m_0^{(1)} + 2m_0^{(2)} + 3m_0^{(\geq 3)} + l, \\ 3n_2 + 2m_0^{(1)} + m_0^{(2)} + 2l - 6 &\geq n_1. \end{aligned}$$

In other terms,

$$\|E\| \leq 3\|F\| + 2m_0^{(1)} + m_0^{(2)} + 2l - 6.$$

Note that $2l - 6 \leq 2c - 6 \leq 0$. Now, estimate the summand $2m_0^{(1)} + m_0^{(2)}$.

Let $M(f)$ be the set of all maximal selected subpaths of the contour of a face f of Δ . Let $M_2(f)$ be the set of all such $p_1 \in M(f)$ that there exist $p_2 \in M(f)$ and a nontrivial path q such that p_1qp_2 is a subpath of the contour of f , and no subpath of q is selected. Let $M_3(f)$ be the set of all such $p_1 \in M(f)$ that there exists $p_2 \in M(f)$ such that p_1p_2 is a non-reduced subpath of the contour of f . The conditions $\mathcal{A}_1(k_1)$, $\mathcal{A}_2(k_2)$, $\mathcal{A}_3(k_3)$ imply that for every $f \in \Delta(2)$,

$$\|M(f)\| \leq k_1(f), \quad \|M_2(f)\| \leq k_2(f), \quad \|M_3(f)\| \leq k_3(f).$$

Let

$$N = \bigcup_{f \in F} M(f), \quad N_2 = \bigcup_{f \in F} M_2(f), \quad N_3 = \bigcup_{f \in F} M_3(f).$$

Define a function $s: N \rightarrow V$ by the following rule: $s(p)$ is the last element of V that the path p meets, i.e., such an element of V that some terminal subpath of p has a vertex in $s(p)$ and has no vertices in any other element of V . It is easy to see that s is well-defined. Assign a weight to every element of N : let the elements of $N_2 \sqcup N_3$ have weight 2, and the other elements of N have weight 1. Let the weight of every subset X of N be the sum of the weights of all its elements. Let the weight of every $v \in V$ be the weight of its full pre-image with respect to s . Let the weight of every $v' \in \Phi(0)$ be the weight of $\alpha_0^{-1}(v')$. It is to be proved now that every vertex of Φ of degree 2 which is not in W' weighs at least 1, and every vertex of Φ of degree 1 which is not in W' weighs at least 2.

Let $v' \in \Phi(0) \setminus W'$ and $v = \alpha_0^{-1}(v')$. View the 2-complex $v \in V$ as a submap of $\bar{\Delta}$. It is a disc map because of the assumptions about E .

Consider the case $d_\Phi(v') = 2$. It is enough to show that the full pre-image of v under s is not empty. Suppose it is empty. Let u_1, u_2 be those oriented arcs from \hat{E} whose terminal vertices are in v (there are exactly 2 such oriented arcs). Let f_i be the face (from F) whose contour has u_i as a subpath, $i = 1, 2$. Then there exist subpaths q_1, q_2 of the contours of f_1, f_2 respectively such that for $i = 1, 2$, the path $u_i q_i u_{3-i}^{-1}$ is a subpath of the contour of f_i . Let q_1, q_2 be the shortest such paths. Note that the cyclic path $q_1 q_2$ is a representative of the contour of the map v . Since the full pre-image of v under s is empty, both paths $u_1 q_1 u_2^{-1}$ and $u_2 q_2 u_1^{-1}$ are selected. Indeed, suppose that for example the path $u_1 q_1 u_2^{-1}$ is not selected. Then there exists a maximal selected subpath p of the contour of f_1 that contains u_1 as a subpath. The path p is an element of $M(f_1)$. Since p cannot contain $u_1 q_1 u_2^{-1}$ as a subpath, it follows that $s(p) = v$, which contradicts to the above assumption. Hence, the paths $u_1 q_1 u_2^{-1}$ and $u_2 q_2 u_1^{-1}$ are selected; therefore, they are reduced.

Suppose the map v is degenerate (for example, consists just of one vertex). Then the paths $u_1 q_1 u_2^{-1}$ and $u_2 q_2 u_1^{-1}$ are mutually inverse. They cannot be oriented arcs because it would contradict to the minimality of the number of elements of U —two arcs corresponding to u_1 and u_2 could be replaced with the one corresponding to $u_1 q_1 u_2^{-1}$. Therefore, $u_1 = u_2^{-1}$, v is the only element of V , and Δ is exceptional. Then N is the full pre-image of v under s , which is empty. Therefore, all subpaths of the contours of f_1 and f_2 are selected. Hence, Δ does not satisfy the condition $\mathcal{Z}(2)$ —not even $\mathcal{Z}(0)$. This gives a contradiction.

Suppose the map v has at least one face. Let S be the set of all nontrivial paths in the set $\{q_1, q_2\}$. The set S consists of selected paths and covers the contour of the map v . Hence, by the Corollary of Proposition 3.1, Δ cannot satisfy the condition $\mathcal{Z}(2)$. Contradiction. Thus, if $d_\Phi(v') = 2$, the weight of v' is at least 1.

Consider the case $d_\Phi(v') = 1$. Let u be the oriented arcs from \hat{E} whose terminal vertex is in v (there is exactly 1 such oriented arc). Let f be the face (from F) whose contour has u as a subpath. Then there exists a path q such that uqu^{-1} is a subpath of the contour of f . Let q be the shortest such path. Note that the cyclic path q is a representative of the contour of the map v .

Suppose the map v is degenerate. Then the path uqu^{-1} is not reduced. Therefore, there exists at least one element of $M_2(f)$ or $M_3(f)$ in the s -pre-image of v , and the weight of v is at least 2.

Suppose the map v has at least one face. Then q is nontrivial. Since Δ satisfies the condition $\mathcal{Z}(2)$, it follows from the Corollary of Proposition 3.1 that the path q is not selected. Moreover, it cannot be the product of two selected subpaths. Therefore, the full pre-image of v under s contains either an element of $M_2(f) \cup M_3(f)$ or at least two elements of $M(f)$. In either case, the weight of v is at least 2.

On one hand, the sum of the weights of all vertices of Φ is at least $2m_0^{(1)} + m_0^{(2)}$; on the other hand, it is not greater than $\|N\| + \|N_2\| + \|N_3\|$. Therefore,

$$2m_0^{(1)} + m_0^{(2)} \leq \sum_{f \in F} (k_1(f) + k_2(f) + k_3(f)).$$

Thus, for an arbitrary nonempty subset E of U ,

$$\begin{aligned} \|E\| &\leq 3\|G(E)\| + \sum_{f \in G(E)} (k_1(f) + k_2(f) + k_3(f)) + 2c - 6 \\ &= \sum_{f \in G(E)} (3 + k_1(f) + k_2(f) + k_3(f)) + 2c - 6. \end{aligned}$$

In particular,

$$\|U\| \leq \sum_{f \in \Delta(2)} (3 + k_1(f) + k_2(f) + k_3(f)) + 2c - 6.$$

By the Corollary of Hall's lemma, there exists a function $h: U \rightarrow \Delta(2)$ such that each $u \in U$ is incident to the face $h(u)$, and the full pre-image of each face $f \in \Delta(2)$ under h consists of at most $3 + k_1(f) + k_2(f) + k_3(f)$ elements. \square

Estimating Lemma 2. *Let Δ be a non-degenerate map. Let S be the set of all (non-ordered) pairs of distinct contiguous faces of Δ . Then $\|S\| < 3\|\Delta(2)\|$, and there exists a function $h: S \rightarrow \Delta(2)$ such that:*

- (1) *each pair $P \in S$ contains the face $h(P)$, and*
- (2) *the full pre-image of each face $f \in \Delta(2)$ under h consists of at most 3 pairs.*

Proof. Define functions $g: S \rightarrow \mathcal{P}(\Delta(2))$ and $G: \mathcal{P}(S) \rightarrow \mathcal{P}(\Delta(2))$ by the formulae:

$$g(P) = P, \quad G(X) = \bigcup X.$$

Take an arbitrary nonempty subset X of S . Let $F = \bigcup X = G(X)$. The set F is nonempty. Let Γ be the contiguity graph for F in Δ . Graph Γ is planar, has no loops and no multiple edges. Obviously, $\|\Gamma(0)\| = \|F\|$ and $\|\Gamma(1)\| \geq \|X\|$. By Proposition 2.1, $\|X\| < 3\|F\|$.

Thus, for any nonempty subset X of S , $\|X\| < 3\|G(X)\|$. In particular,

$$\|S\| < 3\|\Delta(2)\|.$$

Now, the conclusion easily follows from the Corollary of Hall's lemma. \square

6. STATEMENT OF THE MAIN THEOREM

Definition. Let Δ be a simple S-map. Let n_1 be the number of edges of Δ , $n_1^{(e)}$ be the number of external edges of Δ , and S be the number of selected external edges of Δ . Let μ be a real number. The S-map Δ is said to satisfy the *condition* $\mathcal{X}(\mu)$ if

$$S \geq n_1 - \mu(2n_1 - n_1^{(e)}) = \|\Delta(1)\| - \mu \sum_{f \in \Delta(2)} d(f).$$

Remark 6.1. In the above notation, if $\mu < 1$ and Δ satisfies the condition $\mathcal{X}(\mu)$, then

$$S \geq n_1 - \mu(2n_1 - n_1^{(e)}) \geq \max \left\{ \left(1 - \frac{\mu}{1 - \mu}\right)n_1, (1 - 2\mu)(2n_1 - n_1^{(e)}) \right\}.$$

Indeed, since $n_1^{(e)} \geq S$ and $\mu < 1$,

$$\begin{aligned} n_1^{(e)} &\geq n_1 - \mu(2n_1 - n_1^{(e)}), \\ n_1 &\geq 2n_1 - n_1^{(e)} - \mu(2n_1 - n_1^{(e)}), \\ n_1 &\geq (1 - \mu)(2n_1 - n_1^{(e)}), \\ \frac{n_1}{1 - \mu} &\geq 2n_1 - n_1^{(e)}. \end{aligned}$$

Therefore,

$$n_1 - \mu(2n_1 - n_1^{(e)}) \geq (1 - 2\mu)(2n_1 - n_1^{(e)})$$

and

$$n_1 - \mu(2n_1 - n_1^{(e)}) \geq \left(1 - \frac{\mu}{1 - \mu}\right)n_1.$$

Remark 6.2. If a simple S-map Δ satisfies the condition $\mathcal{X}(\mu)$ for some $\mu < \frac{1}{2}$, then Δ is not spherical.

Main Theorem. *Let Δ be a simple S-map with at most 3 contours. Let*

$$k_1, k_2, k_3: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4: \Delta(2) \rightarrow [0, 1],$$

$$\bar{a} = (k_1, k_2, k_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Let μ be a real number. Suppose Δ satisfies the condition $\mathcal{A}(\bar{a})$. Suppose

$$2 \cdot \max_{\Delta(2)} (\lambda_1 + (3 + k_1 + k_2 + k_3)\lambda_2 + 3\lambda_3) + \max_{\Delta(2)} ((2 + k_1)\lambda_2 + 2\lambda_4) < 1,$$

$$\max_{\Delta(2)} (\lambda_1 + (3 + k_1 + k_2 + k_3)\lambda_2 + 3\lambda_3) \leq \mu.$$

Then the S-map Δ satisfies the conditions $\mathcal{X}(\mu)$ and $\mathcal{Z}(2)$.

The main idea of the proof is to verify the conditions $\mathcal{X}(\mu)$ and $\mathcal{Z}(2)$ by simultaneous induction on the number of internal edges of Δ . Necessary inductive lemmas are proved in the next two sections.

Remark 6.3. Provided this theorem holds, the condition $\mathcal{A}(\bar{a})$ and the inequalities in the hypotheses of the theorem may be replaced with the conjunction of only five conditions $\mathcal{A}_1(k_1)$, $\mathcal{A}_4(\lambda_1)$, $\mathcal{A}_5(\lambda_2)$, $\mathcal{A}_6(\lambda_2, \lambda_3)$, $\mathcal{A}_7(\lambda_2, \lambda_4)$, and with the inequalities

$$2 \cdot \max_{\Delta(2)} (\lambda_1 + (3 + 2k_1)\lambda_2 + 3\lambda_3) + \max_{\Delta(2)} ((2 + k_1)\lambda_2 + 2\lambda_4) < 1,$$

$$\max_{\Delta(2)} (\lambda_1 + (3 + 2k_1)\lambda_2 + 3\lambda_3) \leq \mu,$$

respectively. Indeed, for any S-map Δ and function $k_1: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}$ such that Δ satisfies $\mathcal{A}_1(k_1)$, there obviously exist $k_2, k_3: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}$ such that $k_2 + k_3 \leq k_1$ point-wise, and Δ satisfies $\mathcal{A}_2(k_2)$, $\mathcal{A}_3(k_3)$. Thus, k_1 may be substituted for $k_2 + k_3$ in the former inequalities since k_2 and k_3 do not appear anywhere else in the statement of the theorem.

Remark 6.4. In the case when all the functions $k_1, k_2, k_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ are constant, the first inequality in the hypotheses is equivalent to

$$2\lambda_1 + (8 + 3k_1 + 2k_2 + 2k_3)\lambda_2 + 6\lambda_3 + 2\lambda_4 < 1,$$

and can be replaced with

$$2\lambda_1 + (8 + 5k_1)\lambda_2 + 6\lambda_3 + 2\lambda_4 < 1.$$

7. INDUCTIVE LEMMA 1

Inductive Lemma 1. *Let Δ be an S-map. Let*

$$k_1: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_2, \lambda_4: \Delta(2) \rightarrow [0, 1].$$

Suppose Δ satisfies the conditions $\mathcal{A}_1(k_1)$ and $\mathcal{A}_7(\lambda_2, \lambda_4)$. Let μ be a real number such that the following inequality holds point-wise (face-wise):

$$2\mu + (2 + k_1)\lambda_2 + 2\lambda_4 < 1.$$

Suppose that every proper simple disc S -submap of Δ satisfies the condition $\mathcal{X}(\mu)$. Then Δ satisfies the condition $\mathcal{Z}(2)$.

Proof. Suppose that such Δ does not satisfy the condition $\mathcal{Z}(2)$. Then in Δ there exists a simple disc submap Δ_0 such that there exists a set of selected paths with at most 2 maximal elements which covers the contour of Δ_0 . Let Δ_0 be such a submap with the minimal possible number of internal edges. View Δ_0 as an S -submap. Let c be the contour of Δ_0 . Let Q be a set consisting either of 1 selected cyclic path q_1 which is a representative of c , or of 2 selected paths q_1, q_2 whose product q_1q_2 is a representative of c (clearly, such Q exists). Let $s = \|Q\|$.

The map Δ_0 is a proper submap of Δ . Indeed, the edges that lie on the contour of Δ_0 are external in Δ_0 but internal in Δ . Hence, by the assumptions, the S -map Δ_0 satisfies the condition $\mathcal{X}(\mu)$.

Let f_0 be a face of Δ_0 which has at least $(1 - 2\mu)d(f_0)$ selected external edges in Δ_0 . Such a face exists, otherwise the number S of selected external edges of Δ_0 would satisfy the inequality

$$S < \sum_{f \in \Delta_0(2)} (1 - 2\mu)d(f),$$

in contradiction with the condition $\mathcal{X}(\mu)$.

Let P be a non-overlapping set of arcs of Δ such that every element of P is a selected external arc of Δ_0 (i.e., external and selected in Δ_0) incident to f_0 and lying on a path from Q , and such that P covers the set of all selected external edges of Δ_0 incident to f_0 ; moreover, let P be such a set with the minimal possible number of elements. For $i = 1, s$, let P_i be the set of those elements of P that lie on q_i . Let $m = \|P\|$, let $m_i = \|P_i\|$, $i = 1, s$. For $i = 1, s$, enumerate all the subpaths of q_i^{-1} which are oriented arcs corresponding to elements of P_i , according to the order in which they appear on q_i^{-1} : $p_1^{(i)}, \dots, p_{m_i}^{(i)}$. Check now that for arbitrary i, j , there is no selected subpath of the contour of f_0 with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$.

Suppose on the contrary that for some $i \in \{1, s\}$ and some $j \in \{n \in \mathbb{N} \mid 1 \leq n \leq m_i - 1\}$, there is a selected subpath of the contour of f_0 with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$. Let $p_j^{(i)}xp_{j+1}^{(i)}$ be the shortest such selected subpath. Then the path x is either selected or trivial; in either case, it is reduced. Let y be the subpath of q_i that starts at the initial vertex of $p_{j+1}^{(i)}$ and ends at the terminal vertex of $p_j^{(i)}$. The path y is reduced as a subpath of the contour of Δ_0 . Clearly, the cyclic path xy is a representative of the contour of some disc submap Φ of the map Δ_0 . Since the path $p_{j+1}^{(i)-1}yp_j^{(i)-1}$ is a subpath of q_i , and the path $p_j^{(i)}xp_{j+1}^{(i)}$ is selected, the path x cannot be inverse to the path y , otherwise it would contradict to the minimality of the number of elements of P . Therefore, the map Φ is non-degenerate.

By Proposition 3.1, there exists a maximal simple disc submap of Φ whose contour may be covered by 1 or 2 paths each of which is a nontrivial subpath of either x or y . Let Φ_1 be such a submap of Φ . The map Φ_1 is also a submap of Δ . The contour of Φ_1 is covered by a set of 1 or 2 selected paths. The map Φ_1 clearly has fewer internal edges than Δ_0 . This contradicts to the choice of Δ_0 —to the minimality of its number of internal edges.

Since for any i and j , no subpath of the contour of f_0 with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$ is selected, the number of maximal selected subpaths of the contour of f_0 is not less than $m_1 - 1 = m - 1$ in the case $s = 1$, and not less than $m_1 + m_2 - 2 = m - 2$ in the case $s = 2$. Hence, in both cases, $m \leq k_1(f_0) + 2$ by the condition $\mathcal{A}_1(k_1)$.

Let

$$S_i = \sum_{u \in P_i} |u| = \sum_{j=1}^{m_i} |p_j^{(i)}|, \quad i = 1, s, \quad S = \sum_{u \in P} |u| = \sum_{i=1}^s S_i.$$

Note that S equals the number of selected external edges of f_0 in Δ_0 . For each $i = 1, s$, the condition $\mathcal{A}_7(\lambda_2, \lambda_4)$ for Δ implies that

$$S_i \leq (m_i \lambda_2(f_0) + \lambda_4(f_0)) d(f_0).$$

Therefore, on one hand,

$$S \leq (m \lambda_2(f_0) + s \lambda_4(f_0)) d(f_0) \leq ((2 + k_1(f_0)) \lambda_2(f_0) + 2 \lambda_4(f_0)) d(f_0),$$

on the other hand,

$$S \geq (1 - 2\mu) d(f_0).$$

Hence,

$$(2 + k_1(f_0)) \lambda_2(f_0) + 2 \lambda_4(f_0) \geq 1 - 2\mu,$$

and this contradicts to the inequality

$$2\mu + (2 + k_1) \lambda_2 + 2 \lambda_4 < 1. \quad \square$$

8. INDUCTIVE LEMMA 2

Inductive Lemma 2. *Let Δ be a simple S -map with at most 3 contours. Let*

$$k_1, k_2, k_3: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3: \Delta(2) \rightarrow [0, 1].$$

Suppose Δ satisfies the conditions $\mathcal{Z}(2)$, $\mathcal{A}_1(k_1)$, $\mathcal{A}_2(k_2)$, $\mathcal{A}_3(k_3)$, $\mathcal{A}_4(\lambda_1)$, $\mathcal{A}_5(\lambda_2)$, $\mathcal{A}_6(\lambda_2, \lambda_3)$. Let μ be a real number such that the following double inequality holds point-wise (face-wise):

$$\lambda_1 + (3 + k_1 + k_2 + k_3) \lambda_2 + 3 \lambda_3 \leq \mu < 1.$$

Then Δ satisfies the condition $\mathcal{X}(\mu)$.

Proof. Let n_1 be the number of edges of Δ , $n_1^{(e)}$ be the number of external edges of Δ , S be the number of selected external edges of Δ . Note that since Δ is simple,

$$\sum_{f \in \Delta(2)} d(f) = 2n_1 - n_1^{(e)}.$$

First, estimate the number of all edges that are neither selected external, nor double-selected internal. Denote this number by S'_1 . Since S'_1 is less than or equal

to the number of non-selected oriented edges of the contours of all the faces of Δ , and Δ satisfies the condition $\mathcal{A}_4(\lambda_1)$,

$$S'_1 \leq \sum_{f \in \Delta(2)} \lambda_1(f) d(f).$$

Second, estimate the number of double-selected internal edges. Denote this number by S'_2 .

Let U be a minimal by the number of elements non-overlapping set of double-selected internal arcs of Δ covering the set of all double-selected edge of Δ . Let for any $f_1, f_2 \in \Delta(2)$, $B(\{f_1, f_2\})$ be the set of all arcs between f_1 and f_2 that are elements of U (if $f_1 = f_2$, then $B(\{f_1, f_2\}) = B(\{f_1\})$ is the set of all elements of U which are intro-facial arcs of f_1). Let M be the set of all such pairs or singletons X of faces of Δ that $B(X) \neq \emptyset$. Then

$$S'_2 = \sum_{u \in U} |u| = \sum_{X \in M} \sum_{u \in B(X)} |u|.$$

Let N be the set of all pairs of distinct contiguous faces of Δ that have at least one double-selected edge between them.

Let $h_1: U \rightarrow \Delta(2)$ and $h_2: N \rightarrow \Delta(2)$ be such functions that

- (1) each arc $u \in U$ is incident to the face $h_1(u)$;
- (2) the full pre-image of each face f of Δ under h_1 consists of at most $3 + k_1(f) + k_2(f) + k_3(f)$ arcs;
- (3) each pair $P \in N$ contains the face $h_2(P)$;
- (4) the full pre-image of each face f of Δ under h_2 consists of at most 3 pairs.

Such a function h_1 exists by Estimating Lemma 1 if U is not empty, and it obviously exists if U is empty. Such a function h_2 exists by Estimating Lemma 2. (The conditions $\mathcal{A}_1(k_1)$, $\mathcal{A}_2(k_2)$, $\mathcal{A}_3(k_3)$, and $\mathcal{Z}(2)$ have been used here.)

Let K be the subset of $\Delta(2) \times \Delta(2)$ consisting of all ordered pairs (f_1, f_2) such that $\{f_1, f_2\} \in M$. Define functions $r_1: K \rightarrow \mathbb{R}$ and $r_2: K \rightarrow \mathbb{R}$ as follows. For any $(f_1, f_2) \in K$, let

$$\begin{aligned} r_1(f_1, f_2) &= \|h_1^{-1}(f_1) \cap B(\{f_1, f_2\})\| \lambda_2(f_1) d(f_1), \\ r_2(f_1, f_2) &= \|h_2^{-1}(f_1) \cap \{\{f_1, f_2\}\}\| \lambda_3(f_1) d(f_1). \end{aligned}$$

Let $r = r_1 + r_2$.

Take arbitrary distinct faces f_1 and f_2 such that $\{f_1, f_2\} \in M$. For the sake of definiteness, assume that $h_2(\{f_1, f_2\}) = f_1$. Then

$$\begin{aligned} & r(f_1, f_2) + r(f_2, f_1) \\ &= (r_1(f_1, f_2) + r_1(f_2, f_1)) + (r_2(f_1, f_2) + r_2(f_2, f_1)) \\ &= \|h_1^{-1}(f_1) \cap B(\{f_1, f_2\})\| \lambda_2(f_1) d(f_1) + \|h_1^{-1}(f_2) \cap B(\{f_1, f_2\})\| \lambda_2(f_2) d(f_2) \\ &\quad + \lambda_3(f_1) d(f_1) \\ &\geq \|B(\{f_1, f_2\})\| \min\{\lambda_2(f_1) d(f_1), \lambda_2(f_2) d(f_2)\} + \lambda_3(f_1) d(f_1). \end{aligned}$$

By the condition $\mathcal{A}_6(\lambda_2, \lambda_3)$,

$$\begin{aligned} \sum_{u \in B(\{f_1, f_2\})} |u| &\leq \|B(\{f_1, f_2\})\| \min\{\lambda_2(f_1)d(f_1), \lambda_2(f_2)d(f_2)\} + \lambda_3(f_1)d(f_1) \\ &\leq r(f_1, f_2) + r(f_2, f_1). \end{aligned}$$

Now, take an arbitrary face f such that $\{f\} \in M$. Then

$$\begin{aligned} r(f, f) &= r_1(f, f) + r_2(f, f) \\ &= \|h_1^{-1}(f) \cap B(\{f\})\| \lambda_2(f)d(f) + 0 \\ &= \|B(\{f\})\| \lambda_2(f)d(f). \end{aligned}$$

By the condition $\mathcal{A}_5(\lambda_2)$,

$$\sum_{u \in B(\{f\})} |u| \leq \|B(\{f\})\| \lambda_2(f)d(f) = r(f, f).$$

Thus,

$$\begin{aligned} S'_2 &= \sum_{u \in U} |u| = \sum_{X \in M} \sum_{u \in B(X)} |u| \\ &\leq \sum_{(f_1, f_2) \in K} r(f_1, f_2) = \sum_{f_1 \in \Delta(2)} \sum_{f_2: (f_1, f_2) \in K} r(f_1, f_2) \\ &= \sum_{f_1 \in \Delta(2)} \left(\sum_{f_2: (f_1, f_2) \in K} r_1(f_1, f_2) + \sum_{f_2: (f_1, f_2) \in K} r_2(f_1, f_2) \right) \\ &= \sum_{f_1 \in \Delta(2)} (\|h_1^{-1}(f_1)\| \lambda_2(f_1)d(f_1) + \|h_2^{-1}(f_1)\| \lambda_3(f_1)d(f_1)) \\ &\leq \sum_{f_1 \in \Delta(2)} \left((3 + k_1(f_1) + k_2(f_1) + k_3(f_1)) \lambda_2(f_1)d(f_1) + 3\lambda_3(f_1)d(f_1) \right) \\ &= \sum_{f \in \Delta(2)} \left((3 + k_1(f) + k_2(f) + k_3(f)) \lambda_2(f) + 3\lambda_3(f) \right) d(f). \end{aligned}$$

Eventually,

$$\begin{aligned} S &= n_1 - S'_1 - S'_2 \\ &\geq n_1 - \sum_{f \in \Delta(2)} \left(\lambda_1(f) + (3 + k_1(f) + k_2(f) + k_3(f)) \lambda_2(f) + 3\lambda_3(f) \right) d(f) \\ &\geq n_1 - \sum_{f \in \Delta(2)} \mu d(f) = n_1 - \mu(2n_1 - n_1^{(e)}), \end{aligned}$$

since $\lambda_1 + (3 + k_1 + k_2 + k_3)\lambda_2 + 3\lambda_3 \leq \mu$. \square

9. PROOF OF THE MAIN THEOREM

Proof of the main theorem. Suppose the theorem is not true. Then let Δ be a simple S-map, let

$$k_1, k_2, k_3: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4: \Delta(2) \rightarrow [0, 1],$$

$$\bar{a} = (k_1, k_2, k_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

and let μ be a real number such that they all together satisfy the hypotheses of the theorem and do not satisfy the conclusion (i.e., provide a counterexample), and such that the number of internal edges of Δ is the minimal possible under this condition (i.e., the theorem holds whenever the S-map has fewer internal edges). Without loss of generality, assume that

$$\max_{\Delta(2)} (\lambda_1 + (3 + k_1 + k_2 + k_3)\lambda_2 + 3\lambda_3) = \mu.$$

Then

$$2\mu + \max_{\Delta(2)} ((2 + k_1)\lambda_2 + 2\lambda_4) < 1.$$

In particular, $\mu < 1$.

The hypotheses of Inductive Lemma 1 hold for Δ , \bar{a} , and μ . Therefore, by Inductive Lemma 1, Δ satisfies the condition $\mathcal{Z}(2)$. Therefore, by Inductive Lemma 2, Δ satisfies the condition $\mathcal{X}(\mu)$. Hence, the theorem holds for Δ , \bar{a} , and μ , and this contradicts to how they were chosen. \square

10. LEMMA ABOUT EXPOSED FACE

Lemma about exposed face. *Let Δ be a simple disc S-map. Let*

$$k_1: \Delta(2) \rightarrow \mathbb{N} \cup \{0\}, \quad \lambda_2, \lambda_4: \Delta(2) \rightarrow [0, 1], \quad \mu \in \mathbb{R}.$$

Suppose Δ satisfies the conditions $\mathcal{Z}(2)$, $\mathcal{A}_1(k_1)$, $\mathcal{A}_7(\lambda_2, \lambda_4)$. Suppose every simple disc S-submap of Δ satisfies the condition $\mathcal{X}(\mu)$. Then there exists a face f in Δ satisfying the following property: if P is a non-overlapping set of selected external arcs incident to f , if P covers the set of all selected external edges incident to f , and if the number of elements of P is the minimal possible under these assumptions, then $\|P\| \leq k_1(f) + 1$ and

$$\sum_{p \in P} |p| \geq \left(1 - 2\mu - (2 + k_1(f))\lambda_2(f) - \lambda_4(f) + \|P\|\lambda_2(f)\right)d(f).$$

(Such a face f may be called “exposed”).

Remark 10.1. It is possible to prove a stronger statement than the one claimed in this lemma. Namely, under the hypotheses of the lemma, either there exists a face f in Δ such that a corresponding set P has at most $k_1(f) + 1$ elements, and the total sum of their lengths is at least $(1 - 2\mu)d(f)$, or there exist at least two distinct “exposed” faces such as in the conclusion of the lemma.

Remark 10.2. If Δ , $\bar{a} = (k_1, k_2, k_3; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and μ satisfy the hypotheses of the main theorem, then it follows from the main theorem that Δ , k_1 , λ_2 , λ_4 , μ satisfy the hypotheses of the lemma about exposed face.

Proof. Case I: there exists a simple disc submap Δ_0 of Δ whose contour is of the form $q_1 q_2$ where q_1 is a selected subpath of the contour of some face f_0 , and q_2 is a subpath of the contour of Δ . Let Δ_0 be a minimal such map (which do not contain any other such map as a submap). View Δ_0 as an S-submap. Let q_1, q_2 and f_0 be as above. It follows from the condition $\mathcal{Z}(2)$ that q_2 is nontrivial.

Pick a face f_1 of Δ_0 which has at least $(1 - 2\mu)d(f_1)$ selected external edges in Δ_0 . Such a face exists because Δ_0 satisfies the condition $\mathcal{X}(\mu)$.

Let P_1 be a non-overlapping set of double-selected arcs between f_1 and f_0 such that P_1 covers the set of all double-selected edges between f_1 and f_0 . Let P_2 be a non-overlapping set of selected external arcs incident to f_1 such that P_2 covers the set of all selected external edge incident to f_1 . Moreover, let P_1 and P_2 be such sets with the minimal possible numbers of elements. Note that all the elements of P_1 lie on the path q_1 and all the element of P_2 lie on the path q_2 . Let $m_1 = \|P_1\|$, $m_2 = \|P_2\|$. Let $P = P_1 \sqcup P_2$. All the elements of P are selected external arcs of Δ_0 incident to f_1 . By the choice of f_1 ,

$$\sum_{p \in P} |p| \geq (1 - 2\mu)d(f_1).$$

The goal is to prove that $m_2 \leq k_1(f_1) + 1$ and

$$\sum_{p \in P_2} |p| \geq \left(1 - 2\mu - (2 + k_1(f_1))\lambda_2(f_1) - \lambda_4(f_1) + m_2\lambda_2(f_1)\right)d(f_1);$$

this will show that the face f_1 is a desired one.

For $i = 1, 2$, if $P_i \neq \emptyset$, then enumerate all the subpaths of q_i^{-1} which are oriented arcs corresponding to elements of P_i , according to the order in which they appear on q_i^{-1} : $p_1^{(i)}, \dots, p_{m_i}^{(i)}$. Check now that for arbitrary i, j , there is no selected subpath of the contour of f_1 with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$.

Suppose that for some $j \in \{n \in \mathbb{N} \mid 1 \leq n < m_1\}$, there is a selected subpath of the contour of f_1 with initial subpath $p_j^{(1)}$ and terminal subpath $p_{j+1}^{(1)}$. Let $p_j^{(1)} x p_{j+1}^{(1)}$ be the shortest such selected subpath. Then the path x is either selected or trivial; in either case, it is reduced. Let y be the subpath of q_1 that starts at the initial vertex of $p_{j+1}^{(1)}$ and ends at the terminal vertex of $p_j^{(1)}$. The path y is reduced as a subpath of the contour of Δ_0 . Note that $p_{j+1}^{(1)-1} y p_j^{(1)-1}$ is a subpath of q_1 . The cyclic path xy is a representative of the contour of some disc submap Φ of Δ . The path x cannot be inverse to y , otherwise it would contradict to the minimality of the number of elements of P_1 . Therefore, the map Φ is non-degenerate. The Corollary of Proposition 3.1 gives a contradiction with the fact that Δ satisfies the condition $\mathcal{Z}(2)$.

Suppose that for some $j \in \{n \in \mathbb{N} \mid 1 \leq n < m_2\}$, there is a selected subpath of the contour of f_1 with initial subpath $p_j^{(2)}$ and terminal subpath $p_{j+1}^{(2)}$. Let $p_j^{(2)} x p_{j+1}^{(2)}$ be the shortest such selected subpath. Since the disc map Δ is simple and the number of elements of P_2 is minimal, the path x cannot be trivial. Therefore, the path x is selected and reduced. Let y be the subpath of q_2 that starts at the initial vertex of $p_{j+1}^{(2)}$ and ends at the terminal vertex of $p_j^{(2)}$. The path y is reduced as a subpath of the contour of Δ_0 . Note that $p_{j+1}^{(2)-1} y p_j^{(2)-1}$ is a subpath of q_2 . The cyclic path xy is a representative of the contour of some disc submap Φ of Δ . The

path x cannot be inverse to y , otherwise it would contradict to the minimality of the number of elements of P_2 . Therefore, the map Φ is non-degenerate. By Proposition 3.1, there exists a simple disc submap Δ_1 of Φ whose contour is covered by a set of 1 or 2 paths, each of which is either a subpath of x or a subpath of y . In any case, this leads to a contradiction either with the condition $\mathcal{Z}(2)$, or with the fact that Δ is simple, or with the choice of Δ_0 (its minimality).

Thus, for arbitrary i and j , there is no selected path with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$. Hence, it follows from the condition $\mathcal{A}_1(k_1)$ that

$$m_1 \leq k_1(f_1) + 1, \quad m_2 \leq k_1(f_1) + 1, \quad m_1 + m_2 \leq k_1(f_1) + 2.$$

By the condition $\mathcal{A}_7(\lambda_2, \lambda_4)$,

$$\begin{aligned} \sum_{p \in P_1} |p| &\leq (m_1 \lambda_2(f_1) + \lambda_4(f_1)) d(f_1) \\ &\leq ((k_1(f_1) + 2 - m_2) \lambda_2(f_1) + \lambda_4(f_1)) d(f_1). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{p \in P_2} |p| &\geq (1 - 2\mu - (k_1(f_1) + 2 - m_2) \lambda_2(f_1) - \lambda_4(f_1)) d(f_1) \\ &= (1 - 2\mu - (2 + k_1(f_1)) \lambda_2(f_1) - \lambda_4(f_1) + m_2 \lambda_2(f_1)) d(f_1). \end{aligned}$$

Case II: there exists no simple disc submap Δ_0 of Δ whose contour would be of the form $q_1 q_2$ where q_1 would be a selected subpath of the contour of some face, and q_2 would be a subpath of the contour of Δ .

Consider an arbitrary face f of Δ . Let P be a non-overlapping set of selected external arcs incident to f such that P covers the set of all selected external edges incident to f , and the number of elements of P is the minimal possible. Then $\|P\| \leq k_1(f) + 1$. Indeed, if P is empty, this inequality is obvious. Suppose P is not empty. For every selected subpath q of the contour of f , at most one element of P lies on q (because the number of elements of P is minimal, Δ is a simple disc map, Δ satisfies $\mathcal{Z}(2)$, and it is not Case I). If the set of all selected subpath of the contour of f has no maximal element, then P consists of one element (because there exists a selected subpath q of the contour of f such that all the elements of P lie on q), and $\|P\| = 1 \leq k_1(f) + 1$. If the set of all selected subpath of the contour of f has at least one maximal element, then every element of P lies on some maximal selected subpath of the contour of f , and $\|P\| \leq k_1(f) < k_1(f) + 1$ (by $\mathcal{A}_1(k_1)$).

Now, pick a face f_0 in Δ which has at least $(1 - 2\mu)d(f_0)$ selected external edges. The face f_0 is a desired one. \square

11. GROUP PRESENTATIONS AND DIAGRAMS

Recall that an (abstract) *group presentation* is an ordered pair $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ where \mathfrak{A} is an arbitrary set, called *alphabet*, and \mathcal{R} is a set of words in the *group alphabet* $\mathfrak{A}^{\pm 1}$. The elements of \mathcal{R} are called *defining words*. The same group presentation may be also written as $\langle \mathfrak{A} \parallel R = 1, R \in \mathcal{R} \rangle$, here the relations $R = 1, R \in \mathcal{R}$, are called

defining relations. It will be assumed in the rest of this paper that \mathcal{R} does not contain the empty word.

A *group word* in the alphabet \mathfrak{A} is a word in the alphabet $\mathfrak{A}^{\pm 1}$. The *inverse* word to a group word W is denoted by W^{-1} . The *m th power* of a group word W for an integer m is denoted by W^m and defined as follows. If m is a positive integer, then W^m is the result of concatenation of m copies of W . If m is a negative integer, then W^m is the result of concatenation of $-m$ copies of W^{-1} . By definition, W^0 is the empty word. If A and B are two group words, then define $A^B = BAB^{-1}$. A group word is called *reduced* if it has no subwords of the form xx^{-1} where $x \in \mathfrak{A}^{\pm 1}$. The reduced word obtained from a group word W by cancelling one-by-one all subwords of the form xx^{-1} , $x \in \mathfrak{A}^{\pm 1}$, is called the *reduced form* of W (it is well-defined). A group word is *cyclically reduced* if it is reduced and its first letter is not inverse to its last one. The empty word is cyclically reduced by definition. Two words are *freely equal* if their reduced forms coincide. A set of words (or relations) is called *symmetrized* if along with every word W it contains all cyclic shifts of W and W^{-1} . Sometimes, the set of defining words of a presentation will be assumed symmetrized. A *cyclic word* is the set of all cyclic shifts of some word. The cyclic word corresponding to a word W (containing the word W) will be denoted by \tilde{W} . The *length* of a cyclic word is the length of an arbitrary representative of it. A word V is a *subword* of a cyclic word \tilde{W} if it is a subword of some representative of \tilde{W} .

Let $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ be a group presentation. Let W be a group word in the alphabet \mathfrak{A} . The relation $W = 1$ is a *consequence* of the system of relations $\{ R = 1 \mid R \in \mathcal{R} \}$ if, roughly speaking, it may be deduced from relations of this system, associativity, and cancellation of inverse letters.

Every presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ defines a group G , and there is a natural function from the set of all group words in the alphabet \mathfrak{A} onto this group. The image of a group word W under this function is trivial in G if and only if the relation $W = 1$ is a consequence of the system of relations $\{ R = 1 \mid R \in \mathcal{R} \}$. The images of the one-letter words associated with the elements of \mathfrak{A} generate the whole of G . Two group words W_1 and W_2 are said to be *equal in the group G* if their images in G are equal, or, equivalently, if the relation $W_1 = W_2$ is a consequence of the relations $R = 1$, $R \in \mathcal{R}$.

A *diagram* over a presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is a map together with a function which labels every oriented edge of this map with a letter from $\mathfrak{A}^{\pm 1}$ so that mutually inverse oriented edges are labelled with mutually inverse group letters, and for every face, the word that “reads” on some representative of its contour is an element of $\mathcal{R}^{\pm 1}$.

In a diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$, let the *label* of an oriented edge be the element of $\mathfrak{A}^{\pm 1}$ that labels this oriented edge, the *label* of a path be the group word that reads on this path, the *label* of a cycle be the cyclic (group) word whose representatives are the labels of representatives of this cycle, the *label* of a face be the label of its contour.

According to van Kampen’s Lemma (see [O]), a relation $W = 1$ is a consequence of a system of relations $\{ R = 1 \mid R \in \mathcal{R} \}$ if and only if there exists a disc diagram Δ such that the label of some representative of its contour is W , and for every face f of Δ , the label of some representative of the contour of f either is an element of \mathcal{R} , or is inverse to some element of \mathcal{R} . Such a diagram is called a *deduction diagram* for the word W or relation $W = 1$.

A pair of distinct faces $\{f_1, f_2\}$ in a diagram Δ is called *cancellable* if there exist paths p_1 and p_2 with common initial oriented edge and with equal labels such that p_1 is a representative of the contour of f_1 , and p_2^{-1} is a representative of the contour of f_2 . A diagram without cancellable pairs of faces is called *reduced*. If a relation $W = 1$ is a consequence of a system of relations $\{R = 1 \mid R \in \mathcal{R}\}$, then, in fact, there exists a reduced deduction diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ for the relation $W = 1$.

Definition. A symmetrized set S of group words in a given alphabet satisfies the *small cancellation condition* $C'(\lambda)$, $\lambda > 0$, if every element of S is a nonempty reduced word, and for any two distinct words $W_1, W_2 \in S$, the length of any common prefix of W_1 and W_2 is less than $\lambda \min\{|W_1|, |W_2|\}$. A symmetrized group presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ is said to satisfy the condition $C'(\lambda)$, $\lambda > 0$, if the set of defining words \mathcal{R} satisfies $C'(\lambda)$.

This and other classical small cancellation conditions and their applications may be found in [LySc].

Remark 11.1. A symmetrized group presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ satisfies the condition $C'(\lambda)$, $\lambda > 0$, if and only if for every reduced simple disc diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$, there exists $\lambda_2 < \lambda$ such that Δ satisfies the condition $\mathcal{A}(0, 0, 0; 0, \lambda_2, 0, 0)$ (all the constants here are regarded as constant functions on $\Delta(2)$).

12. PROOF OF THEOREM 1

Proof of Theorem 1 (see Introduction). Let F_n be a free group of rank n , $n \in \mathbb{N}$. Let $\mathfrak{B} = \{x_1, \dots, x_n\}$ be a basis of F_n . Let λ be a positive number less than $1/13$. Let S be an infinite subset of F_n such that all the elements of S are cyclically reduced relative to \mathfrak{B} and are not proper powers in F_n . Suppose that the symmetrization of S relative to \mathfrak{B} satisfies the small cancellation condition $C'(\lambda)$ relative to \mathfrak{B} . The goal is to prove that there exists an infinite simple 2-generated group P and a homomorphism $\phi: F_n \rightarrow P$ such that ϕ maps S surjectively onto P .

Let S' be an infinite subset of S which satisfies the following three properties if the elements of F_n are regarded as reduced group words in the alphabet \mathfrak{B} :

- (1) If $w_1 s v_1$ or $(w_1 s v_1)^{-1}$ belongs to S' , and $w_2 s v_2$ or $(w_2 s v_2)^{-1}$ belongs to S' , then either $w_1 = w_2$ and $v_1 = v_2$, or $|s| \leq \lambda \min\{|w_1 s v_1|, |w_2 s v_2|\}$. (Here, the phrase “ $w_1 s v_1$ belongs to S' ” means that the concatenation of the group words w_1, s, v_1 is a reduced group word representing an element of S' .)
- (2) If u_1 and u_2 are elements of S' , then $u_1 \neq u_2^{-1}$.
- (3) If u is an element of S' , then $|u| \geq 5$.

Such a set S' may be obtained from S by first leaving out all the elements of length less than 5 and then picking one representative out of each of the equivalence classes of the following equivalence relation: call elements u_1 and u_2 equivalent if u_2 is a cyclic shift of u_1 or u_1^{-1} . Note that condition (1) implies that if $u \in S'$, then any common subword of u and u^{-1} is of length at most $\lambda|u|$ (though, it also follows from the condition $C'(\lambda)$ for S).

Let

$$\lambda_1 = \frac{5}{13} - 5\lambda, \quad \lambda_2 = \lambda, \quad \mu = \frac{5}{13}.$$

Then $\lambda_1, \lambda_2 \in [0, 1]$ and

$$\begin{aligned} 2\lambda_1 + 13\lambda_2 &< 1, \\ \lambda_1 + 5\lambda_2 &= \mu. \end{aligned}$$

Let F_{n+2} be a free group of rank $n + 2$ with a basis $\mathfrak{A} = \{x_1, \dots, x_n, a, b\}$, containing F_n as a subgroup. Note that F_{n+2} is the inner free product of the subgroup F_n and the subgroup generated by $\{a, b\}$. In the rest of this proof, all the elements of F_{n+2} shall be regarded as reduced group words in the alphabet \mathfrak{A} . The product of two elements of F_{n+2} shall mean the concatenation of the group words; it may be not reduced. The usual group multiplication will not be used.

Split the set S' into the disjoint union of two infinite sets S_1 and S_2 . The first step of “constructing” the group P consists in imposing three systems of relations on the group F_{n+2} . The first system of relations uses elements of S_1 and ensures that every element of the obtained quotient group G is represented by some element of S_1 (in particular, the quotient homomorphism maps F_n onto G); the second system uses elements of S_2 and ensures that the quotient group G has no nontrivial finite homomorphic images; the third system ensures that the elements of G represented by a and b generate the whole of G . The main theorem shall be used for proving that the quotient group G is not trivial. The second step of constructing P consists in taking the quotient of G over its maximal proper normal subgroup.

Let $v_{1,1}, v_{1,2}, v_{1,3}, \dots$ be a list of all reduced group words in the alphabet \mathfrak{A} . Using infiniteness of S_1 and finiteness of \mathfrak{B} , it is easy to show that there exists a system $\{u_{1,i}\}_{i=1}^{+\infty}$ of pairwise distinct elements of S_1 such that $\lambda_1|u_{1,i}| \geq |v_{1,i}|$ for every $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, let

$$r_{1,i} = u_{1,i}v_{1,i}^{-1}.$$

(Here, $r_{1,i}$ is the concatenation of $u_{1,i}$ and $v_{1,i}^{-1}$; it does not need to be reduced.) Let $\mathcal{R}_1 = \{r_{1,i} \mid i = 1, 2, \dots\}$.

Let a function from a set X to a group G be called *trivial* if it maps all the elements of X to the neutral element of G . Let M be a set of nontrivial finite groups such that every nontrivial finite group is isomorphic to exactly one group in M . Such a set M exists and is countable. Informally speaking, M is a set of up to isomorphism all nontrivial finite groups. Let T be the set of all ordered pairs (G, ψ) where $G \in M$, and ψ is a nontrivial function from \mathfrak{A} to G . Clearly, T is countable. Let $(G_1, \psi_1), (G_2, \psi_2), \dots$ be a list of all elements of T (without recurrences). Let $u_{2,1}, u_{2,2}, u_{2,3}, \dots$ be a list (without recurrences) of elements of S_2 of length at least $1/\lambda_1$. For every $i \in \mathbb{N}$, let $v_{2,i}$ be a group word over \mathfrak{A} of minimal length such that the values of $v_{2,i}$ and $u_{2,i}$ in G_i with respect to ψ_i are not equal. Clearly, the length of every such $v_{2,i}$ is either 0 or 1. For every $i \in \mathbb{N}$, let

$$r_{2,i} = u_{2,i}v_{2,i}^{-1}.$$

Let $\mathcal{R}_2 = \{r_{2,i} \mid i = 1, 2, \dots\}$.

Let $S_3 = \{u_{3,1}, \dots, u_{3,n}\}$ be a set consisting of n reduced group words of length at least $1/\lambda_1$ in the alphabet $\{a, b\}$ which satisfies the same three conditions as those required of S' above. (Such a set S_3 exists.) For every $i \in \{1, \dots, n\}$, let $v_{3,i} = x_i$ and

$$r_{3,i} = u_{3,i}v_{3,i}^{-1}.$$

Let $\mathcal{R}_3 = \{ r_{3,i} \mid i = 1, \dots, n \}$.

Let

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3.$$

Let G be the group defined by the presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$. This group may be naturally identified with a quotient group of F_{n+2} . Let $\phi_1: F_{n+2} \rightarrow G$ be the corresponding epimorphism.

Because of the relations $r = 1, r \in \mathcal{R}_1$, the homomorphism ϕ_1 maps S_1 onto G .

Because of the relations $r = 1, r \in \mathcal{R}_2$, the group G has no nontrivial finite quotients. Indeed, suppose G has a nontrivial finite quotient. Then for some i there exists an epimorphism ϕ_2 from G onto G_i such that $\phi_2 \circ \phi_1$ extends ψ_i . On one hand, the value of the word $r_{2,i}$ in G_i with respect to ψ_i is nontrivial by the construction of $r_{2,i}$. On the other hand, the value of $r_{2,i}$ in G with respect to ϕ_1 is 1 since $r_{2,i} \in \mathcal{R}$; therefore, the value of $r_{2,i}$ in G_i with respect to $\phi_2 \circ \phi_1$ is 1. This gives a contradiction.

Because of the relations $r = 1, r \in \mathcal{R}_3$, the group G is generated by $\phi_1(a)$ and $\phi_1(b)$.

The only property of G that still needs to be established, is that G is nontrivial.

Suppose G is trivial. Then, by van Kampen's Lemma, for any group word v in the alphabet \mathfrak{A} , there exists a disc diagram Δ over the presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ such that the label of some representative of the contour of Δ is v . In particular, there exists a reduced disc diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ whose contour has length 1. Let Δ by such a diagram. Clearly, it is simple.

Define a selection on Δ as follows. Consider all the faces of Δ one-by-one. Take an arbitrary face f of Δ on which the selection has not been defined yet. Let i and j be such indices that the label of some representative of the contour of f is either $r_{i,j}$ or $r_{i,j}^{-1}$. Then there exist paths p and q such that pq is a representative of the contour of f , and either the label of p is $u_{i,j}$, the label of q is $v_{i,j}^{-1}$, or the label of p is $u_{i,j}^{-1}$, the label of q is $v_{i,j}$. In both cases, such a path p is reduced since the word $u_{i,j}$ is reduced. Let the selection on f consist of all the nontrivial subpaths of this p . The diagram Δ has become a *diagram with selection* or an *S-diagram*.

Now, check that the S-diagram Δ satisfies the condition $\mathcal{A}(1, 1, 1; \lambda_1, \lambda_2, 0, 0)$ (here, all the constants shall be interpreted as constant functions on $\Delta(2)$).

The condition $\mathcal{A}_1(1)$ follows directly from the construction of the selection. The conditions $\mathcal{A}_2(1)$ and $\mathcal{A}_3(1)$ follow from $\mathcal{A}_1(1)$ (see Remark 4.1).

In order to check the condition $\mathcal{A}_4(\lambda_1)$, consider an arbitrary face f and the corresponding subpaths p and q of its contour used above in defining the selection on f . Note that p is the only maximal selected subpath of the contour of f , and q is the “complement” of p in the contour of f . Take i and j such that p is labelled by $u_{i,j}$ or $u_{i,j}^{-1}$, and q is labelled by $v_{i,j}^{-1}$ or $v_{i,j}$. By the construction of the words $u_{i,j}$ and $v_{i,j}$, the inequality $\lambda_1 |u_{i,j}| \geq |v_{i,j}|$ holds. Therefore,

$$|q| \leq \lambda_1 |p| \leq \lambda_1 |pq| = \lambda_1 d(f).$$

The condition $\mathcal{A}_4(\lambda_1)$ follows.

To check the conditions $\mathcal{A}_5(\lambda_2)$, $\mathcal{A}_6(\lambda_2, 0)$, $\mathcal{A}_7(\lambda_2, 0)$, it is enough to prove that for all faces f of Δ , the length of every double-selected oriented arc t which is a subpath of the contour of f is at most $\lambda_2 d(f)$.

Consider first an arbitrary intro-facial double-selected oriented arc t . Let f be the face whose contour has t as a subpath. Let p be the maximal selected subpath of the contour of f , and q be such that pq is a representative of the contour of f (the same notation as above). Let u be the label of p . By the choice of the selection, either u or u^{-1} belongs to $S' \cup S_3$. The contour of f has the form $ts_1t^{-1}s_2$. Since t and t^{-1} are selected, and there is exactly one maximal selected subpath of the contour of f , either ts_1t^{-1} or $t^{-1}s_2t$ is selected. So, either ts_1t^{-1} or $t^{-1}s_2t$ is a subpath of p . In either case, the label of t is a common subword of u and u^{-1} . Therefore,

$$|t| \leq \lambda|p| \leq \lambda d(f) = \lambda_2 d(f)$$

(see property (1) of S' and S_3).

Consider next an arbitrary inter-facial double-selected oriented arc t . Let f_1 be the face whose contour has t as a subpath, and f_2 be the face whose contour has t^{-1} as a subpath (t is between f_1 and f_2). Let p_1 be the maximal selected subpath of the contour of f_1 , and p_2 be the maximal selected subpath of the contour of f_2 . By the choice of the selection, $|p_1| \leq d(f_1)$ and $|p_2| \leq d(f_2)$. Let u_1 be the label of p_1 , and u_2 be the label of p_2 . One of the words u_1, u_1^{-1} and one of the words u_2, u_2^{-1} belong to $S' \cup S_3$. The label of t is a common subword of u_1 and u_2^{-1} . If $|t| > \lambda|p_1|$ or $|t| > \lambda|p_2|$, then the pair of faces $\{f_1, f_2\}$ is cancellable (follows from the properties of S' and S_3 , the fact that S' and S_3 use disjoint alphabets, and the construction of \mathcal{R}). Since Δ is reduced, this cannot happen. Therefore,

$$|t| \leq \lambda|p_1| \leq \lambda d(f_1) = \lambda_2 d(f_1).$$

Thus, Δ satisfies the conditions $\mathcal{A}_5(\lambda_2), \mathcal{A}_6(\lambda_2, 0), \mathcal{A}_7(\lambda_2, 0)$.

Since Δ satisfies the condition $\mathcal{A}(1, 1, 1; \lambda_1, \lambda_2, 0, 0)$, by the main theorem and subsequent Remark 6.3, the length of its contour is at least $(1 - 2\mu) \sum_{f \in \Delta(2)} d(f)$. Since

$$\begin{aligned} (1 - 2\mu) \sum_{f \in \Delta(2)} d(f) &\geq (1 - 2\mu) \min\{|r| \mid r \in \mathcal{R}\} \\ &\geq (1 - 2\mu) \min\{|u| \mid u \in S_1 \cup S_2 \cup S_3\} \geq \frac{3}{13} \cdot 5 > 1, \end{aligned}$$

this contradicts to the assumption that the length of the contour of Δ is 1. Hence, the group G could not be trivial.

By Zorn's Lemma, there exists a maximal proper normal subgroup N of the group G since G is finitely generated. Let $P = G/N$. Let $\phi_2: G \rightarrow P$ be the quotient homomorphism. Then P is a desired infinite simple 2-generated group, and $\phi = \phi_2 \circ \phi_1|_{F_n}$ is a desired epimorphism $F_n \rightarrow P$. \square

Proof of the Corollary of Theorem 1 (see Introduction). Take a free group F of rank 27 with a basis $\{a_1, \dots, a_{27}\}$. Let

$$S = \{a_1^m \cdots a_{27}^m \mid m \in \mathbb{N}\}.$$

The symmetrization of the set S obviously satisfies the condition $C'(\frac{2}{27} + \varepsilon)$ relative to the basis $\{a_1, \dots, a_{27}\}$ for any $\varepsilon > 0$. Therefore, by Theorem 1, there exists an infinite simple 2-generated group G and a homomorphism $\phi: F \rightarrow G$ such that ϕ maps S onto G . Let $x_i = \phi(a_i)$, $i = 1, \dots, 27$. \square

13. PROOF OF THEOREM 2

Proof of Theorem 2 (see Introduction). In this proof, let

$$k_1 = 28, \quad \lambda_1 = 10^{-2}, \quad \lambda_2 = 10^{-4}, \quad \lambda_4 = \frac{1}{28}, \quad \lambda_3 = 4\lambda_4 = \frac{1}{7}.$$

Let $\mu = 0.45$. These numbers satisfy the inequalities

$$\begin{aligned} 2\lambda_1 + (8 + 5k_1)\lambda_2 + 6\lambda_3 + 2\lambda_4 &< 1, \\ \lambda_1 + (3 + 2k_1)\lambda_2 + 3\lambda_3 &\leq \mu. \end{aligned}$$

Let $\bar{a} = (k_1, k_1, k_1; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Let $\mathfrak{A} = \{x, y\}$ be a 2-letter alphabet. Let $(a_1, b_1), (a_2, b_2), \dots$ be all the ordered pairs of reduced group words in the alphabet \mathfrak{A} listed without recurrences. Let, moreover, the function $i \mapsto (a_i, b_i)$ be computable.

Pick two nonempty reduced words v_1, v_2 and a system of nonempty reduced words $\{u_{i,j}\}_{i=1,2,\dots, j=1,\dots,14}$ so that the following conditions are satisfied:

- (1) If $i \in \mathbb{N}$, then $|u_{i,1}| = |u_{i,2}| = \dots = |u_{i,14}|$.
- (2) If $i \in \mathbb{N}$, then $28|u_{i+1,1}| + 14|a_{i+1}| + |b_{i+1}| > 28|u_{i,1}| + 14|a_i| + |b_i|$.
- (3) If $i \in \mathbb{N}$, then $28|u_{i,1}| \geq 99(14|a_i| + |b_i|)$.
- (4) If $i_1, i_2 \in \mathbb{N}$, $j_1, j_2 \in \{1, 2, 3, \dots, 14\}$, $\sigma_1, \sigma_2 \in \{\pm 1\}$, $u_{i_1, j_1}^{\sigma_1} = w_1 s z_1$ and $u_{i_2, j_2}^{\sigma_2} = w_2 s z_2$, then either $w_1 = w_2$, $z_1 = z_2$, $i_1 = i_2$, $j_1 = j_2$, $\sigma_1 = \sigma_2$, or $|s| \leq 28 \cdot 10^{-4} \min\{|u_{i_1, j_1}|, |u_{i_2, j_2}|\}$.
- (5) The products (i.e., concatenations) $v_1 v_1, v_2 v_2, v_1 v_2, v_1^{-1} v_2, v_1 v_2^{-1}, v_1^{-1} v_2^{-1}$ are reduced. For example, v_1 starts and ends with the letter x , and v_2 starts and ends with the letter y .
- (6) If $v \in \{v_1^{\pm 1}, v_2^{\pm 1}\}$ and $u \in \{u_{i,j}^{\pm 1} \mid i \in \mathbb{N}, j = 1, \dots, 14\}$, then $|u| \geq 40|v|$ and v is not a subword of u .

Besides that, assume that the function $(i, j) \mapsto u_{i,j}$ is computable. Such words v_1, v_2 and a system $\{u_{i,j}\}_{i=1,2,\dots, j=1,\dots,14}$ exist.

For every $i = 1, 2, 3, \dots$, let

$$r_i = a_i^{u_{i,1}} a_i^{u_{i,2}} \dots a_i^{u_{i,14}} b_i^{-1}$$

(recall that $a_i^{u_{i,j}} = u_{i,j} a_i u_{i,j}^{-1}$). Note that by condition (2),

$$|r_1| < |r_2| < |r_3| < \dots,$$

by condition (3), for every i ,

$$14|a_i| + |b_i| \leq \lambda_1 |r_i|,$$

by condition (1), for every i ,

$$|u_i| \leq \lambda_4 |r_i|.$$

Define a sequence of sets $\{\mathcal{R}_i\}_{i=0}^{+\infty}$ inductively. Let $\mathcal{R}_0 = \emptyset$. Take $i > 0$. If the relation $a_i = 1$ is a consequence of the relations $r = 1$, $r \in \mathcal{R}_{i-1}$ (for example, if a_i is the empty word), then let $\mathcal{R}_i = \mathcal{R}_{i-1}$. Otherwise, let $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{r_i\}$. Let

$$\mathcal{R} = \bigcup_{i=1}^{+\infty} \mathcal{R}_i.$$

Let G be the group defined by the presentation $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$. Let ϕ be the natural function from the set of all group words in the alphabet \mathfrak{A} onto G . Let H be the subgroup of G generated by $\phi(v_1)$ and $\phi(v_2)$. It easily follows from the imposed relations that G is 14-boundedly simple. Moreover, for any nontrivial element $g \in G$ and for any element $h \in G$, the element h is the product of 14 conjugates of g . The next goal is to prove that the word problem in G is solvable and that H is freely generated by $\phi(v_1)$ and $\phi(v_2)$. It will be done using the fact that every reduced simple disc diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ satisfies the condition $\mathcal{A}(\bar{a})$ (where the components of \bar{a} are viewed as constant functions on $\Delta(2)$).

In the rest of this proof, a selection on a diagram Δ over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ will be called *special* if for every face f of Δ , there is a natural number i , a representative c of the contour of f , and paths $q_0, \dots, q_{14}, p_1, \dots, p_{14}, p'_1, \dots, p'_{14}$ such that the following conditions hold:

- the label of c is either r_i or r_i^{-1} ;
- if the label of c is r_i , then

$$c = p'_1 q_1 p_1^{-1} p'_2 q_2 p_2^{-1} \cdots p'_{14} q_{14} p_{14}^{-1} q_0^{-1},$$

and the selection on f consists of all nontrivial subpaths of $p_1^{-1}, \dots, p_{14}^{-1}$ and p'_1, \dots, p'_{14} ;

- if the label of c is r_i^{-1} , then

$$c = q_0 p_{14} q_{14}^{-1} p_{14}'^{-1} p_{13} q_{13}^{-1} p_{13}'^{-1} \cdots p_1 q_1^{-1} p_1'^{-1},$$

and the selection on f consists of all nontrivial subpaths of $p_1'^{-1}, \dots, p_{14}'^{-1}$ and p_1, \dots, p_{14} .

- the label of q_0 is b_i , and for every $j = 1, \dots, 14$, the labels of p_j and p'_j are $u_{i,j}$, the label of q_j is a_i .

Clearly, on every diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ there exists a special selection. A diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ together with a special selection will be called a *special S-diagram*. Note that if a simple disc S-map Δ' is cut out from a special S-diagram, and the S-map Δ' is naturally labelled via some pasting morphism, then the S-diagram obtained from Δ' is special.

Consider an arbitrary special S-diagram Δ . Clearly, Δ satisfies the conditions $\mathcal{A}_1(k_1)$, $\mathcal{A}_2(k_1)$, and $\mathcal{A}_3(k_1)$. It easily follows from condition (3) that Δ also satisfies $\mathcal{A}_4(\lambda_1)$. It is to be shown that in fact every reduced special disc S-diagram satisfies $\mathcal{A}(\bar{a})$.

Use induction on the number of internal edges of a reduced special disc S-diagram. If a reduced special S-diagram has no internal edges, then it clearly satisfies the condition $\mathcal{A}(\bar{a})$. Suppose that Δ is a reduced special disc S-diagram,

and every reduced special disc S-diagram having fewer internal edges than Δ satisfies the condition $\mathcal{A}(\bar{a})$. To complete the inductive proof, it is enough to prove now that Δ satisfies $\mathcal{A}(\bar{a})$. Since it is enough to prove that every maximal simple disc S-subdiagram of Δ satisfies $\mathcal{A}(\bar{a})$, assume that Δ is simple.

The length of every selected subpath of the contour of every face f of Δ is at most $\lambda_4 d(f)$. Consider arbitrary (not necessarily distinct) faces f_1 and f_2 of Δ . Suppose there exists a double-selected arc t between f_1 and f_2 such that $|t| > \lambda_2 d(f_1)$ (call such arcs “long”). Then, by condition (4), the labels of f_1 and f_2 are either equal or mutually inverse. Since the diagram Δ is reduced, the labels of f_1 and f_2 in fact are equal. Hence, the contours of f_1 and f_2 have the same length. Therefore, if p is a selected subpath of the contour of f_2 on which a given “long” arc lies, then the sum of the lengths of all maximal double-selected arcs between f_1 and f_2 that lie on p is at most $\lambda_4 d(f_1)$. Suppose, moreover, that the faces f_1 and f_2 are distinct. Then it follows from the choice of the selection and from the properties of the system $\{u_{i,j}\}_{i=1,2,\dots, j=1,\dots,14}$ that all such “long” double-selected arcs between f_1 and f_2 cannot be situated on more than 4 maximal selected subpaths of the contour of f_1 (as well as of f_2). Therefore, the sum of the lengths of all such “long” maximal double-selected arcs between f_1 and f_2 is not greater than $4\lambda_4 d(f_1) = \lambda_3 d(f_1)$. Thus, Δ satisfies the conditions $\mathcal{A}_7(\lambda_2, \lambda_4)$ and $\mathcal{A}_6(\lambda_2, \lambda_3)$.

Verification of the condition $\mathcal{A}_5(\lambda_2)$ is more complicated. Suppose that $\mathcal{A}_5(\lambda_2)$ does not hold in Δ . Let f_0 be a face of Δ and t be a double-selected inward intro-facial oriented arc such that t is a subpath of the contour of f_0 and $|t| > \lambda_2 d(f_0)$ (such f_0 and t exist since $\mathcal{A}_5(\lambda_2)$ does not hold). Let s_1 and s_2 be such subpaths of the contour of f_0 that $ts_1t^{-1}s_2$ is a representative of the contour of f_0 . Since every reduced special disc S-diagram having fewer internal edges than Δ satisfies the condition $\mathcal{A}_5(\lambda_2)$, the choice of f_0 and t is in fact unique and the path s_2^{-1} is a representative of the contour of Δ (otherwise the cycle represented by s_2^{-1} would cut out an S-diagram having fewer internal edges and still not satisfying $\mathcal{A}_5(\lambda_2)$).

Let i be the number such that $d(f_0) = |r_i|$. Then the label of some representative of the contour of f_0 is either r_i or r_i^{-1} . By the construction of r_i (see condition (4)) and by the definition of a special selection, there is exactly one $m \in \{1, 2, \dots, 14\}$ and one $\sigma \in \{\pm 1\}$ such that the label of t is a subword of $u_{i,m}^\sigma$. Then either the label of s_1 or the label of $t^{-1}s_2t$ has the form

$$(wa_iw^{-1})^{\pm 1},$$

where w is a suffix of $u_{i,m}$.

Case 1: the label of s_1 has the form

$$(wa_iw^{-1})^{\pm 1},$$

w is a suffix of $u_{i,m}$. Let Δ_0 be the disc subdiagram of Δ with the contour represented by s_1 . Note that $\Delta(2) = \Delta_0(2) \cup \{f_0\}$. By the construction of $\{\mathcal{R}_i\}_{i=0}^{+\infty}$, the label of s_1 is not trivial modulo the set of relation

$$\{r = 1 \mid r \in \mathcal{R}_{i-1}\} = \{r = 1 \mid r \in \mathcal{R}, |r| < d(f_0)\}.$$

Pick a face f_1 in Δ_0 such that $d(f_1) \geq d(f_0)$. Let Δ_1 be the maximal simple disc S-subdiagram of Δ containing the face f_1 and not containing the face f_0 . All the

external edges of Δ_1 lie on the path s_1 . In particular, the length of the contour of Δ_1 is at most $|s_1|$. The S-map Δ_1 satisfies $\mathcal{A}_5(\lambda_2)$ since it has fewer internal edges than Δ . Hence, by the main theorem and subsequent Remark 6.3, the length of the contour of Δ_1 is at least $(1 - 2\mu)d(f_1)$. Therefore, on one hand,

$$|s_1| \geq (1 - 2\mu)d(f_1) \geq (1 - 2\mu)d(f_0) = \frac{1}{10}d(f_0).$$

On the other hand,

$$|s_1| < |a_i| + 2|u_{i,m}| \leq \frac{1}{14}d(f_0).$$

Contradiction.

Case 2: the label of $t^{-1}s_2t$ has the form

$$(wa_iw^{-1})^{\pm 1},$$

w is a suffix of $u_{i,m}$. Let Δ' be an S-diagram cut out of Δ by the cycle represented by $t^{-1}s_2^{-1}t$. (Informally speaking, the S-diagram Δ' may be thought of as obtained from Δ by cutting Δ along t from the boundary inside.) The S-diagram Δ' is a simple reduced special disc S-diagram, it has fewer internal edges than Δ . Therefore, it satisfies $\mathcal{A}_5(\lambda_2)$. Hence, by the main theorem and subsequent Remark 6.3, the length of the contour of Δ' is at least $(1 - 2\mu)d(f'_0)$ where f'_0 is the face of Δ' corresponding to f_0 (note that $d(f'_0) = d(f_0)$). Therefore, on one hand,

$$|t^{-1}s_2t| \geq (1 - 2\mu)d(f_0) = \frac{1}{10}d(f_0).$$

On the other hand,

$$|t^{-1}s_2t| \leq |a_i| + 2|u_{i,m}| \leq \frac{1}{14}d(f_0).$$

Contradiction.

Thus, every reduced special disc S-diagram Δ satisfies $\mathcal{A}(\bar{a})$. In particular, Δ , \bar{a} , and μ satisfy the assumptions of the main theorem. Hence, if n is the length of the contour of Δ , then the sum of the lengths of the contours of all the faces of Δ is at most $\frac{1}{1-2\mu}n$, and the number of edges of Δ is at most $\frac{1-\mu}{1-2\mu}n$.

Now, if Δ is an arbitrary reduced disc diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$, the length of the contour of Δ is greater than or equal to the sum of the lengths of the contours of all its maximal simple disc subdiagrams. Hence, if n is the length of the contour of Δ , then again the sum of the lengths of the contours of all the faces of Δ is at most $\frac{1}{1-2\mu}n$, and the number of edges of Δ is at most $\frac{1-\mu}{1-2\mu}n$.

For every $i = 0, 1, 2, \dots$, let G_i be the group defined by the presentation $\langle \mathfrak{A} \parallel \mathcal{R}_i \rangle$, and let ϕ_i be the natural function from the set of all group words in the alphabet \mathfrak{A} onto G_i .

Consider an arbitrary group word w whose value in G is 1. Let Δ be a reduced deduction diagram for w over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$. Let $n = |w|$. Consider an arbitrary $i > 10n$. Then $|r_i| > i > 10n = (1 - 2\mu)^{-1}n$. So, no face in Δ is labelled with a cyclic word associated with r_i or r_i^{-1} . Hence, the value of w in G_{10n} is 1 as well.

Recall that an algorithm solves the word problem in $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ if for every group word w in the alphabet \mathfrak{A} , it determines whether or not w is trivial in G , i.e.,

whether the relation $w = 1$ is a consequence of the relations $r = 1, r \in \mathcal{R}$. Here is an informal description of an algorithm that solves the word problem in $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$.

Consider group words only in the alphabet \mathfrak{A} . Let **Alg_1** be an algorithm that for every finite set \mathcal{S} of group words and for every group word w decides whether there exists a deduction diagram Δ for w over $\langle \mathfrak{A} \parallel \mathcal{S} \rangle$ with at most $\frac{1-\mu}{1-2\mu}|w|$ edges. Clearly, such an algorithm exists. Let **Alg_2** be an algorithm that for every natural i constructs the set \mathcal{R}_i . It starts with $\mathcal{R}_0 = \emptyset$, and then for every $l = 1, \dots, i$ constructs \mathcal{R}_l by using **Alg_1** to check whether $\mathcal{R}_l = \mathcal{R}_{l-1}$ or $\mathcal{R}_l = \mathcal{R}_{l-1} \cup \{r_l\}$. Let **Alg_3** be an algorithm that for every group word w first uses **Alg_2** to construct \mathcal{R}_{10n} where $n = |w|$, and then uses **Alg_1** to determine whether there exists a deduction diagram Δ for w over $\langle \mathfrak{A} \parallel \mathcal{R}_{10n} \rangle$ with at most $\frac{1-\mu}{1-2\mu}n$ edges. Such a diagram exists if and only if w is trivial in G_{10n} . Hence, the algorithm **Alg_3** solves the word problem in $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$.

It is left to prove that H is freely generated by $\phi(v_1)$ and $\phi(v_2)$. Suppose it is not. Then there exist $m \geq 1, i_1, \dots, i_m \in \{1, 2\}, \sigma_1, \dots, \sigma_m \in \{\pm 1\}$ such that

$$(\forall j \in \{1, \dots, m-1\})(i_j \neq i_{j+1} \text{ or } \sigma_j = \sigma_{j+1}) \text{ and } (i_m \neq i_1 \text{ or } \sigma_m = \sigma_1),$$

and the group word $w = v_{i_1}^{\sigma_1} \dots v_{i_m}^{\sigma_m}$ is trivial in G . By the choice of v_1 and v_2 , the group word w is cyclically reduced. Consider an arbitrary reduced deduction diagram Δ for w over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$. The contour of Δ is reduced since w is cyclically reduced. Let Δ_0 be a maximal simple disc submap of Δ such that some representative of the contour Δ_0 is a subpath of the contour of Δ . Let q_0 be a representative of the contour of Δ_0 which is a subpath of the contour of Δ . Define a selection on Δ_0 the same way as above (special selection). As it has been shown, the obtained S-diagram satisfies $\mathcal{A}(\bar{a})$. Apply the main theorem with subsequent Remark 6.3 and the lemma about exposed face to Δ_0 . Let f be a face of Δ_0 and P be a non-overlapping set of selected external arcs of Δ_0 incident to f such that $\|P\| \leq k_1 + 1$ and

$$\sum_{p \in P} |p| \geq (1 - 2\mu - (2 + k_1)\lambda_2 - \lambda_4 + \|P\|\lambda_2)d(f).$$

Note that elements of P are arcs in Δ_0 but do not need to be arcs in Δ since some intermediate vertex of some element of P may have degree greater than 2 in Δ . Let P_0 be the set of all maximal elements of the set of all arcs that lie on q_0 and are subarcs of elements of P . Then P_0 is a non-overlapping set of arcs, $\|P_0\| \leq \|P\| + 1$, and $\sum_{p \in P_0} |p| = \sum_{p \in P} |p|$. Let p be a longest arc in P_0 . Then

$$\begin{aligned} |p| &\geq \frac{1 - 2\mu - (2 + k_1)\lambda_2 - \lambda_4 + \|P\|\lambda_2}{\|P_0\|}d(f) \\ &\geq \frac{1 - 2\mu - (2 + k_1)\lambda_2 - \lambda_4 + (\|P_0\| - 1)\lambda_2}{\|P_0\|}d(f) \\ &= \frac{1 - 2\mu - (3 + k_1)\lambda_2 - \lambda_4}{\|P_0\|}d(f) + \lambda_2d(f). \end{aligned}$$

The picked values of $k_1, \lambda_2, \lambda_4$ and μ satisfy the inequality

$$1 - 2\mu - (3 + k_1)\lambda_2 - \lambda_4 \geq 0.$$

Therefore,

$$\begin{aligned} |p| &\geq \frac{1 - 2\mu - (3 + k_1)\lambda_2 - \lambda_4}{k_1 + 2} d(f) + \lambda_2 d(f) \\ &= \frac{1 - 2\mu - \lambda_2 - \lambda_4}{k_1 + 2} d(f). \end{aligned}$$

Therefore, $|p| > \frac{1}{500} d(f)$. Let p' be the subpath of the contour of f corresponding to the arc p (so, p' is a selected oriented arc of Δ). Let i be the number such that $d(f) = |r_i|$. Take $j \in \{1, 2, \dots, 14\}$ and $\sigma \in \{\pm 1\}$ such that the label of p' is a subword of $u_{i,j}^\sigma$. Since $28|u_{i,j}| < d(f)$, it follows that

$$|p'| > \frac{1}{20} |u_{i,j}| \geq 2 \max\{|v_1|, |v_2|\}.$$

Since p'^{-1} is a subpath of the contour of Δ , and $|p'| \geq 2 \max\{|v_1|, |v_2|\}$, at least one of the words $v_1^{\pm 1}$, $v_2^{\pm 1}$ is a subword of the label of p' , and therefore, it is a subword of $u_{i,j}^\sigma$. Contradiction with condition (5). Thus, H is freely generated by $\phi(v_1)$ and $\phi(v_2)$. The proof is complete. \square

The author would like to thank Alexander Ol'shanskii for suggesting the problems and a possible approach to their solution.

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VANDERBILT UNIVERSITY, DEPT OF MATHEMATICS
 1326 STEVENSON CENTER
 NASHVILLE TN 37240-0001
E-mail address: alexey.muranov@vanderbilt.edu